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Towards Algebraic descriptions of Formal Concept and Rough Set Analysis

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Abstract.
Formal concept analysis (FCA) as introduced in [4] deals with contexts and concepts. Roughly speaking, a context is an environment that is equipped with some kind of "knowledge". Such contexts are also known as information or knowledge representation systems where the knowledge consists of (intensional) descriptions relating sets of objects to sets of properties. Given extsensional and intensional descriptions (the latter one in terms of binary attributes), they can be arranged in a taxonomy or concept lattice.

Rough set theory (RST) or rough set data analysis (RSDA), [17], is a method to describe arbitrary sets of objects in terms of logic expressions based on many-valued attributes. Given an arbitrary set and a partition on the domain, the lower approximation is the union of all equivalence classes that are included; its upper approximation consists of all objects whose classes have a common element with the set. The prime application of RST is to identify minimal sets of features (many-valued attributes) such that the intersection of the induced equivalence relations allows a sufficiently close approximation.

Obviously, both approaches have a strong lattice theoretic background but instead of embedding RST and FCA into each other via lattice theory, we show their mutual inclusion algebraically.
The core construction used are residuals and their interpretation as maximal preconditions satisfying domain set inclusion.

1 Foundations

Throughout the chapter we shall use the standard notation for set theory calculus. Concerning lattice theory, we will use the squarish operators and relation symbols $\sqcap, \sqcup, \sqsubseteq$ unless we speak of algebras of sets or relations. Sets and types are denoted by lowercase letters r, s, t, relations by uppercase letters P, Q, R. Domain object variables are written as x, y, z. By $\mathcal{U}$ we refer to the universe, i.e. the set of all objects under consideration. Sets of sets, in particular sets of relations or functions, are typeset in boldface characters $R \in \mathbf{R} \subseteq \mathbb{R}$ and $f \in \mathbf{F} \subseteq \mathbb{F}$ (attributes are written $a \in \mathbf{A} \subseteq \mathbb{A}$). Complements, converse and composition are indicated by $\overline{s}$ or $\overline{R}$, $R^{\smile}$ and $P \mathbin{;} Q$. Set difference is written as $s - t$, quotients (partitions) as s/R (and $s\backslash R$ where applicable), equivalence

classes as $[x]_R$, residuals as $P\backslash\!\!\backslash Q$ and $P/\!\!/Q$. (Pre-) images are usually denoted by $s.P$ and $P.t$; for reasons that shall become clear later, we write $|P\rangle\, s$ and $\langle R|\, s$ instead. To indicate the "direction" of a relation we write $R : s \rightharpoondown t$ rather than $R \subseteq s \times t$. The universal relation is $\top\!\!\!\top_{(s,t)} = s \times t$ and the empty relation $\perp\!\!\!\perp_{(s,t)} = \emptyset$; 1_s is the (sub-) identity relation on $s \times s$ (if $s \subseteq \mathcal{U}$). The characteristic function of a set s is denoted $\dot{s} : \mathcal{U} \to \mathbf{2}$ and $\widetilde{f}$ denotes the equivalence induced by a function f.[1]
Proofs of relevant theorems are given in the appendix so as not to disrupt the flow of reading. The appendix also contains a brief summary of notational conventions and an index.

1.1 Information systems

Formal concept analysis (FCA) as introduced in [4] deals with structured *contexts*. Roughly speaking, a *context* is an environment that is equipped with some kind of "knowledge". Such contexts are also known as *information* or *knowledge representation systems* where the knowledge consists of descriptions relating sets of objects to sets of properties. In its most general form, such a system can be defined as a domain set $\mathcal{U}$ with a set $\mathbb{F}$ of functions each of which assigns a distinct value of its codomain to an object:

$$\mathfrak{I} = \langle \mathcal{U}, \mathbb{F}, V_{\mathbb{F}} \rangle\,. \tag{1}$$

Such systems are usually represented as tables with a row for each element $x \in \mathcal{U}$ and a column for each feature $f \in \mathbb{F}$ and the value $f(x)$ in the x-row and f-column. We assume every $f \in \mathbb{F}$ to be total by assigning x a value $f(x) = ?_f$ should it be undefined.
Each information system $\mathfrak{I}$ comes with an information relation (or "*query*" function) $\mathtt{Q}$:

$$\mathtt{Q} : \mathcal{U} \times (\mathcal{U} \to V_{\mathbb{F}}) \to V_{\mathbb{F}} \text{ with } \langle x, f\rangle\, \mathtt{Q} f(x) \Longleftrightarrow \mathtt{Q}(x, f) = f(x) \in V_f\,. \tag{2}$$

Binary features $f : \mathcal{U} \rightharpoondown \mathbf{2}$ are called *attributes*. We then say that the infix-predicate $\mathtt{P}$ is true for its arguments, iff the corresponding query $\mathtt{Q}$ delivers $\mathbf{1}$:

$$x\mathtt{P}f :\Longleftrightarrow \langle x, f\rangle\, \mathtt{Q}\ \mathbf{1} \Longleftrightarrow f(x) = \mathbf{1}. \tag{3}$$

Assuming that V_f is finite for every $f \in \mathbb{F}$, we can transform $\mathfrak{I}$ into an information system $\mathfrak{J} = \langle \mathcal{U}, \mathbb{G}, \mathbf{2}_{\mathbb{G}} \rangle$ with $\mathbb{G}$ containing only attributes f_v by *fanning-out* all features:

$$f_v : \mathcal{U} \to \mathbf{2} \text{ with } f_v(x) = \mathbf{1} :\Longleftrightarrow f(x) = v \tag{4}$$

for all $f \in \mathbb{F}$ and for all $v \in V_f$. The result of an attribute-based information system generated by fanning-out a feature-based one is a sparse relation. Figure

[1] For the sake of readability, singleton sets $s = \{x\}$ may be written as x when clear from context; stacked operators maybe reduced to the topmost operator when clear from context ($\widetilde{s} = \widetilde{\dot{s}}$). When clear from context, we may generalize from or switch between equivalent notations. For example, $s = 1_s = \langle \dot{s}|\,\mathbf{1} = S$.

	$\mathbb{F}$			
$\mathfrak{I}$	*col*	*shp*	*edg*	*siz*
□	`w`	`square`	4	`S`
■	`b`	`square`	4	`B`
▪	`b`	`square`	4	`S`
●	`g`	`circle`	1	`S`
△	`w`	`triangle`	3	`B`
◆	`b`	`diamond`	4	`S`
○	`w`	`circle`	1	`S`

	$\mathbb{A}$											
	col			*shp*				*edg*			*siz*	
$\mathfrak{I}$	`w`	`g`	`b`	`c`	`t`	`d`	`s`	1	3	4	`S`	`B`
□	1	0	0	0	0	0	1	0	0	1	1	0
■	0	0	1	0	0	0	1	0	0	1	0	1
▪	0	0	1	0	0	0	1	0	0	1	1	0
●	0	1	0	1	0	0	0	1	0	0	1	0
△	1	0	0	0	1	0	0	0	1	0	0	1
◆	0	0	1	0	0	1	0	0	0	1	1	0
○	1	0	0	1	0	0	0	1	0	0	1	0

Fig. 1. An information system; feature-based (l) and attribute-based (r).

1 displays an information system containing knowledge about geometric figures over the domain $\mathcal{U} = \{$□, ■, ▪, ●, △, ◆, ○$\}$.[2]

A *formal context* is a structure $\mathfrak{K} = \langle \mathcal{U}, \mathbb{F}, \mathfrak{I} \rangle$ where $\mathcal{U}$ is a collection of objects, $\mathbb{F}$ is a set of features f and $\mathfrak{I}$ an information system $\langle \mathcal{U}, \mathbb{F}, V_{\mathbb{F}} \rangle$ defining `Q`. Usually, all features in a formal context are assumed to be attributes; but by fanning out we can transform any $\mathbb{F}$ into an equivalent $\mathbb{A}$ such that we can always construct $\mathfrak{K} = \langle \mathcal{U}, \mathbb{A}, \mathfrak{I} \rangle$ with `P` from $\mathfrak{I} = \langle \mathcal{U}, \mathbb{F}, V_{\mathbb{F}} \rangle$.

Any (common-sense) concept can be defined in two ways: either by listing all its instances or by providing a law by which one can decide whether an arbitrary object is an instance of this concept or not. Hence, a concept is an element of $\wp(\mathcal{U}) \times \wp(\mathbb{A})$. For a concept $\mathbf{c} = \langle s, \mathbf{A} \rangle$, $s = ext(\mathbf{c})$ is called $\mathbf{c}$'s *extension* and $\mathbf{A} = int(\mathbf{c})$ its *intension*.

1.2 Domain operators

The usual operators

$$|R\rangle\, s := s.R = \{y : \ \exists x : x \in s \wedge xRy\} \tag{5}$$

$$\langle R|\, s := R.s = \{x : \ \exists y : x \in s \wedge xRy\} \tag{6}$$

where $\langle R|\, s = |R^{\smile}\rangle\, s$ are not sufficient to describe commonly shared attributes of a collection of objects (or vice versa). [4] introduced the "$'$"-operators that are still standard in FCA, but also noted their sometimes insufficient clarity, which is why we will use a *strict* (pre-) image operator: For a binary relation R we

[2] For the ease of writing we have chosen all feature value sets in this example to be pairwise disjoint. Hence, we will use the abbreviation $v(x) := f_v(x)$ in this example for the remainder of the article.

define

$$|R]\, s := \bigcap_{x \in s} x.R = \{y \in \mathcal{U} : \forall x : x \in s \longrightarrow xRy\} = \bigcap_{x \in s} |R\rangle\, \{x\} \tag{7}$$

$$[R|\, t := \bigcap_{y \in t} R.y = \{x \in \mathcal{U} : \forall y : y \in t \longrightarrow xRy\} = \bigcap_{y \in t} \langle R|\, \{y\} \tag{8}$$

with s and t being subsets of R's domain and codomain.[3] Even though we have chosen the (strict) domain operators in allusion to modal logic operators, they are *not* dual w.r.t. their image but rather R:

$$\overline{|R\rangle\, s} = |\overline{R}]\, s \text{ and } \overline{|R]\, s} = |\overline{R}\rangle\, s. \tag{9}$$

Proof (Equation 9). Pointwise, by logic and set theory:

$$\begin{aligned}
\overline{|R\rangle\, s} &\overset{|\ \rangle}{=\!=} \overline{\{y :\ \exists x : x \in s \wedge xRy\}} \overset{\text{deM}}{=\!=} \overline{\left\{y :\ \exists x : \neg(x \notin s \vee x\overline{R}y)\right\}} \\
&\overset{\neg,\overline{\ }}{=\!=} \left\{y :\ \neg\exists x : \neg(x \notin s \vee x\overline{R}y)\right\} \overset{\neg\exists\neg}{=\!=} \left\{y :\ \forall x : (x \notin s \vee x\overline{R}y)\right\} \\
&=\!= \left\{y :\ \forall x : x \in s \longrightarrow x\overline{R}y\right\} \overset{|\ ]}{=\!=} |\overline{R}]\, s.
\end{aligned}$$

□

However, we will find this "odd" duality to be rather natural later. On the other hand, there is a natural subsumption relation

$$\overline{|R\rangle\, s} \subseteq |R]\, \overline{s} \text{ and } \overline{|R]\, s} \subseteq |R\rangle\, \overline{s}. \tag{10}$$

Proof (Equation 10).

$$\begin{aligned}
\overline{|R\rangle\, s} &= \overline{\{y :\ \exists x : x \in s \longrightarrow xRy\}} \\
&= \{y :\ \neg\exists x : x \in s \longrightarrow xRy\} \\
&= \{y :\ \forall x : \neg(x \in s \longrightarrow xRy)\} \\
&= \{y :\ \forall x : \neg(x \notin s \vee xRy)\} \\
&= \{y :\ \forall x : x \in s \wedge xRy\} \\
&\subseteq \{y :\ \forall x : x \in s \vee xRy\} \\
&= \{y :\ \forall x : x \notin s \longrightarrow xRy\} \\
&= \bigcap_{x \in \overline{s}} |R\rangle\, x \\
&= |R]\, \overline{s}.
\end{aligned}$$

The reverse direction follows from antitony of $\overline{\ }$ w.r.t. $\subseteq$. □

[3] As usual, we abbreviate the application of domain operators on singleton sets by dropping the set braces; i.e. $X.\, \{x\} = X.x$ for an arbitrary domain operator $X.$.

This already gives rise to our later formalisation using Galois connections. Trivially, we have

$$|R]\, s \overset{\text{Def}}{=\!=} \bigcap_{x\in s} |R\rangle\, \{x\} \overset{\cup/\cap}{\subseteq} \bigcup_{x\in s} |R\rangle\, \{x\} \overset{\text{iso}}{\subseteq} \bigcup_{x\in s} |R\rangle\, s = |R\rangle\, s \tag{11}$$

(and the same for preimages), but in contrast to $|\ \rangle$, we also have

$$s \subseteq [R|\, |R]\, s. \tag{12}$$

Proof (Equation 12). Let $x \in s$. We then have to show that $x \in [R|\, (|R]\, s)$ which is equivalent to xRy for all $y \in |R]\, s$. By (7) and strictness of $|\ ]$, $y \in s\, |R]\, s$ if zRy for all $z \in s$, and hence for $x \in s$ in particular which is true by assumption. □

In addition to this, we discover the following antitone behaviour. Let s, s' and t, t' be subsets of R's domain and codomain. Then,

$$s' \subseteq s \Longrightarrow |R]\, s \subseteq |R]\, s' \text{ and } t' \subseteq t \Longrightarrow [R|\, t \subseteq [R|\, t'. \tag{13}$$

Proof (Equation 13). $|\ ]$ and $[\ |$ are $\subseteq$-antitone.
(1) $s' \subseteq s \Longrightarrow |R]\, s \subseteq |R]\, s'$: (a) We derive a contradiction by assuming $s' \subseteq s$ and $|R]\, s \not\subseteq |R]\, s'$. (b) The latter is equivalent to writing $\exists y \in \mathcal{U}' : y \in |R]\, s \wedge y \notin |R]\, s'$. (c) From $y \in |R]\, s$ and the definition of $|\ ]$ it follows that $\forall x \in s : xRy$. (d) On the other hand, we have $s' \subseteq s$ which means $\forall x \in s' : xRy$, too. (e) Then, again, we know that $y \in |R]\, s'$ which contradicts (b).
(2) $t' \subseteq t \Longrightarrow [R|\, t \subseteq [R|\, t'$: Since $|R]\, t = \bigcap \{x\}_{.}\, R = \bigcap R^{\smile}{}_{.}\, \{x\} = |R^{\smile}]\, t$, we rewrite $[R|\, t'$ as $|R^{\smile}]\, t'$ and apply (1). □

A very important property of $|\ ]$ and $[\ |$ is the following:

$$|R]\, s = |R]\, [R|\, |R]\, s \text{ and } [R|\, t = [R|\, |R]\, [R|\, t \tag{14}$$

but *not* $s = [R|\, |R]\, s$ or $t = |R]\, [R|\, t$ unless R is an equivalence.

Proof (Equation 14). The equality is shown by mutual inclusion. First,

$$\begin{aligned} & |R]\, s \subseteq |R]\, [R|\, |R]\, s \\ & |\text{: Renaming } t := |R]\, s \\ \Longleftrightarrow\ & t \subseteq |R]\, [R|\, t. \end{aligned}$$

which is true by equation (12). By (12) again and (13),

$$s \subseteq [R|\, |R]\, s \Longrightarrow |R]\, [R|\, |R]\, s \subseteq |R]\, s.$$

The second equality follows by replacement of R with $R^{\smile}$. □

The interplay between $|\ ]$ and $|\ \rangle$ can also be expressed by the following inequality:

$$[R|\, |R\rangle\, s \subseteq [R|\, |R]\, s \subseteq \langle R|\, |R]\, s \subseteq \langle R|\, |R\rangle\, s \text{ and the same for } R^{\smile}. \tag{15}$$

Proof (Equation 15). Since $|R]\,s \subseteq |R\rangle\,s$ and $[R|\,s \subseteq \langle R|\,s$, the first inclusion follows from antitony of $[\,|$ and restricting $|\,\rangle$ to $|\,]$. The second inclusion follows from isotony of $\langle\,|$ and the third one from weakening $|\,]$ to $|\,\rangle$ (see the set inclusion in equation 11). In detail:

$$[R|\,|R\rangle\,s = [R|\bigcup_{x\in s}|R\rangle\,x \;\subseteq\; [R|\bigcap_{x\in s}|R\rangle\,x \;=\; [R|\,|R]\,s \tag{16}$$

$$= \bigcap_{y\in|R]\,s}|R\rangle\,y \subseteq \bigcup_{y\in|R]\,s}|R\rangle\,y \;=\; \langle R|\,|R]\,s \;\subseteq\; \langle R|\,(\,|R\rangle\,s).\Box \tag{17}$$

For the special case of R being an equivalence we not for later purposes:

$$\begin{aligned} s \subseteq |R\rangle\,s {=} \langle R|\,|R\rangle\,s {=} |R\rangle\,\langle R|\,s {=} \langle R|\,s \\ = |R]\,s {=} [R|\,|R]\,s {=} |R]\,[R|\,s {=} \langle R|\,s \subseteq s. \end{aligned} \tag{18}$$

1.3 Galois connectedness

We have made several observations indicating that the constructions presented so far have a strong flavour of Galois connections. Actually, it holds that

$$s \subseteq [R|\,t \Longleftrightarrow t \subseteq |R]\,s. \tag{19}$$

Proof (Equation 19). Assume $s \subseteq [R|\,t$. Then, by equation (12), $t \subseteq |R]\,[R|\,t$. By our assumption and due to the antitony of $[\,|$ as in equation (13), it follows that $|R]\,[R|\,t \subseteq |R]\,s$. The same argument applies to the reverse direction. Hence, we have

$$s \overset{(12)}{\subseteq} [R|\,|R]\,s \overset{(13)}{\subseteq} [R|\,t \text{ and} t \overset{(12)}{\subseteq} |R]\,[R|\,t \overset{(13)}{\subseteq} |R]\,s.\Box \tag{20}$$

Thus, $|\,]$ forms the left adjoint to $[\,|$ and without further effort, we can conclude

$$s \subseteq [R|\,|R]\,s \text{ and } \;|R]\,[R|\,t \subseteq t \tag{21}$$

which proves equation 12; also, equation (14) follows directly from the laws of Galois connections. Finally, by coincidence, the composition of domain operation $[\,|\,|\,]$ is called *extensive* (and, as we shall see, it points out to "extensions").

2 Formal concept analysis: pairing objects and attributes

We now examine the case where R is formal context's information relation $\mathtt{P} : \mathcal{U} \rightharpoondown \mathbb{A}$ and use the (pre-) image operations for reasining about intents and extents.

$\mathtt{P}$	a	b	c	d	e
v		**1**	**1**		
w		**1**			**1**
x		**1**		**1**	
y		**1**	**1**	**1**	**1**
z	**1**		**1**	**1**	**1**

For $s = \{w, x, y\}$ and $\mathbf{A} = \{b, c, d\}$, $\mathbf{c} = \langle s, \mathbf{A} \rangle$ is a preconcept, because

$$\begin{aligned} |\mathtt{P}]\, s &= \{b, e\} \cap \{b, d\} \cap \{b, c, d, e\} \\ &= \{b\} \subseteq \mathbf{A} \\ [\mathtt{P}|\, \mathbf{A} &= \{v, w, x, y\} \cap \{v, y, z\} \cap \{x, y, z\} \\ &= \{y\} \subseteq s. \end{aligned}$$

$\mathtt{P}'$	a	b	c	d	e
v	**1**		**1**		
w		**1**	**1**	**1**	**1**
x		**1**	**1**	**1**	
y	**1**				**1**
z					**1**

Fig. 2. A preconcept (left) and a concept (right).

2.1 Concepts

A preconcept is a pair of intension and extension that are mutually included by their corresponding strict (pre-) images:

$$\mathbf{c} = \langle s, \mathbf{A} \rangle \text{ is a } \textit{preconcept}, \text{ iff } s \subseteq [\mathtt{P}|\, \mathbf{A} \text{ and } \mathbf{A} \subseteq |\mathtt{P}]\, s. \tag{22}$$

An example is shown in figure 2. If one inequality can be turned into full equality, $\mathbf{c}$ is called a *semiconcept*; when both are equalities, $\mathbf{c}$ is a *concept*. In this case, the matrix can be arranged in a way such that $\mathbf{c}$ forms a *rectangle* in $\mathtt{P}$ (see the right part in figure 2). Intensions and extensions of concepts are (Galois–) connected through $|\ ]$ as already suggested by the antitone behaviour shown in equation (13) an in section 1.3.[4] Also, for any $s, t \subseteq \mathcal{U}$ and $\mathbf{A}, \mathbf{B} \subseteq \mathbb{A}$,

$$s \subseteq t \Longrightarrow |\mathtt{P}]\, t \subseteq |\mathtt{P}]\, t \text{ and } \mathbf{A} \subseteq \mathbf{B} \Longrightarrow [\mathtt{P}|\, \mathbf{B} \subseteq [\mathtt{P}|\, \mathbf{A}. \tag{23}$$

It follows that for $\mathbf{c} = \langle s, \mathbf{A} \rangle$,

$$\mathbf{c} \text{ is a preconcept} \Longrightarrow [\mathtt{P}|\, |\mathtt{P}]\, s \subseteq s \tag{24}$$

$$\mathbf{c} \text{ is a } \quad \text{concept} \Longrightarrow [\mathtt{P}|\, |\mathtt{P}]\, s = s \tag{25}$$

and the same for $[\mathtt{P}|\, \mathbf{A}$ — but not for $|\ \rangle$.
The definition of *concepts* is, in fact, very strict. If $\mathbf{c} = \langle s, \mathbf{A} \rangle$ is a concept,

$$[\mathtt{P}|\, |\mathtt{P}]\, s = [\mathtt{P}|\, |\mathtt{P}]\, \mathit{ext}(\mathbf{c}) = [\mathtt{P}|\, \mathbf{A} = [\mathtt{P}|\, \mathit{int}(\mathbf{c}) = s \tag{26}$$

and $|\mathtt{P}]\, [\mathtt{P}|\, \mathbf{A} = \mathbf{A}$.
Let $\mathit{Con}(\mathfrak{K}) \subseteq \wp(\mathcal{U}) \times \wp(\mathbb{F})$ denote the set of all concepts in a context $\mathfrak{K} = \langle \mathcal{U}, \mathbb{A}, \mathfrak{I} \rangle$. Trivially, both universe and feature powersets form lattices, and, hence, their product forms a product lattice with a partial order $\sqsubseteq = \subseteq \times \supseteq$ (see, for example, [6, 7]). With $\mathit{Con}(\mathfrak{K})$ being a subset, construction of a lattice requires a partial order $\sqsubseteq$ defined by meet and join operations based on $|\ ]$ such that

$$\mathbf{c} = \langle s, \mathbf{A} \rangle \sqsubseteq \mathbf{d} = \langle t, \mathbf{B} \rangle :\Longleftrightarrow s \subseteq t \Longleftrightarrow \mathbf{B} \subseteq \mathbf{A}, \tag{27}$$

[4] [4]'s proposition 11 follows directly from this observation. Note also, that $|\mathtt{P}\rangle\, g$ is isotone: $s \subseteq t \Longrightarrow |R\rangle\, s \subseteq |R\rangle\, t$.

where $\mathbf{c}$ and $\mathbf{d}$ are called *sub-* and *superconcepts*, respectively. This is achieved by defining

$$\bigsqcap Con(\mathfrak{K}) := \left\langle \bigcap S, |\mathtt{P}] \, [\mathtt{P}| \bigcup A \right\rangle \text{ and} \tag{28}$$

$$\bigsqcup Con(\mathfrak{K}) := \left\langle [\mathtt{P}| \, |\mathtt{P}] \bigcup S, \bigcap A \right\rangle \tag{29}$$

where $S = \{ext(\mathbf{c}) : \mathbf{c} \in Con(\mathfrak{K})\}$ and $A = \{int(\mathbf{c}) : \mathbf{c} \in Con(\mathfrak{K})\}$. It forms a complete lattice with[5]

$$\mathbf{c} \sqsubseteq \mathbf{d} :\Longleftrightarrow \mathbf{c} \sqcup \mathbf{d} = \mathbf{d} \Longleftrightarrow \mathbf{c} \sqcap \mathbf{d} = \mathbf{c}. \tag{30}$$

A pointwise proof delivers a precise definition of $\bot_{\mathfrak{K}}$ and $\top_{\mathfrak{K}}$ but is quite tesious and error prone. Therefore, we also give a concise, pointfree proof alternative.

Proof (Equations 28, 29). $\bigsqcup$ and $\bigsqcap$ are $\sqsubseteq$-supremum and infimum operators. We show this by proving that every two-element subset has an infimum and supremum.
We show that $\mathbf{c} \sqcap \mathbf{d}$ is the greatest lower bound of $\mathbf{c}$ and $\mathbf{d}$ for arbitrary $\mathbf{c}, \mathbf{d} \in Con(\mathfrak{K})$. Since $\mathcal{U}$ and $\mathbb{A}$ are finite, the product of their respective powersets is finite, too. Therefore, we show that there are unique greatest and smallest elements $\top_{\mathfrak{K}}, \bot_{\mathfrak{K}}$ which together shows that $\langle Con(\mathfrak{K}), \sqcap, \top_{\mathfrak{K}}, \bot_{\mathfrak{K}} \rangle$ forms a complete lattice. A detailed proof for $\sqcap$ being the dual of $\sqcup$ is omitted. Let $\mathbf{c} = \langle s, \mathbf{A} \rangle$ and $\mathbf{d} = \langle t, \mathbf{B} \rangle$ and $\mathbf{c} \sqcap \mathbf{d} = \langle s \cap t, |\mathtt{P}] \, [\mathtt{P}| \, \mathbf{A} \cup \mathbf{B} \rangle$.
(1) We first show that $\mathbf{c} \sqcap \mathbf{d} \in Con(\mathfrak{K})$. With $\mathbf{c}, \mathbf{d}$ being concepts, it holds that

$$\begin{aligned} & int(\mathbf{c}) {=} \mathbf{A} {=} |\mathtt{P}] \, s, \; ext(\mathbf{c}) {=} s {=} [\mathtt{P}| \, \mathbf{A} \\ \text{and} \;\; & int(\mathbf{d}) {=} \mathbf{B} {=} |\mathtt{P}] \, t, \; ext(\mathbf{d}) {=} t {=} [\mathtt{P}| \, \mathbf{B}. \end{aligned}$$

We then have to show that

$$[\mathtt{P}| \, |\mathtt{P}] \, (s \cap t) = s \cap t \text{ and } |\mathtt{P}] \, [\mathtt{P}| \, |\mathtt{P}] \, [\mathtt{P}| \, (\mathbf{A} \cup \mathbf{B}) = |\mathtt{P}] \, [\mathtt{P}| \, (\mathbf{A} \cup \mathbf{B}),$$

where the second equation can be reduced to $|\mathtt{P}] \, [\mathtt{P}| \, (\mathbf{A} \cup \mathbf{B}) = \mathbf{A} \cup \mathbf{B}$. We derive

$$\begin{aligned} & s \cap t \\ & |\text{: by assumption } \mathbf{c}, \mathbf{d} \in Con(\mathfrak{K}) \\ = \; & [\mathtt{P}| \, \mathbf{A} \cap [\mathtt{P}| \, \mathbf{B} \\ = \; & \bigcap_{a \in \mathbf{A}} \langle \mathtt{P}| \, a \cap \bigcap_{b \in \mathbf{B}} \langle \mathtt{P}| \, b \\ = \; & \bigcap_{f \in \mathbf{A} \cup \mathbf{B}} \langle \mathtt{P}| \, f = [\mathtt{P}| \, \mathbf{A} \cup \mathbf{B} \end{aligned}
\qquad
\begin{aligned} & \mathbf{A} \cup \mathbf{B} \\ & |\text{: by assumption } \mathbf{c}, \mathbf{d} \in Con(\mathfrak{K}) \\ = \; & |\mathtt{P}] \, s \cup |\mathtt{P}] \, t \\ = \; & \bigcap_{x \in s} |\mathtt{P}\rangle \, x \cup \bigcap_{y \in t} |\mathtt{P}\rangle \, y \\ = \; & \bigcap_{z \in s \cup t} |\mathtt{P}\rangle \, z = |\mathtt{P}] \, s \cup t \end{aligned}$$

[5] Also, the concept lattices for two contexts $\mathfrak{K} = \left\langle \mathcal{U}, \mathbb{F}, \mathtt{P}^{\smile} \right\rangle$ and $\mathfrak{K}^{\smile} = \langle \mathbb{F}, \mathcal{U}, \mathtt{P} \rangle$ are isomorphic by simply exchanging $\mathbf{F}$ and $\mathcal{U}$ and mapping every concept $\mathbf{c} = \langle s, \mathbf{F} \rangle \in \mathfrak{K}$ onto $\mathbf{c}^{\smile} = \langle \mathbf{F}, s \rangle \in \mathfrak{K}^{\smile}$ which inherits $\sqsubseteq$ by the second equivalence in equation (27).

$\mathfrak{I}$	w	g	b	c	t	d	s	1	3	4	S	B
	A											
	col			*shp*				*edg*			*siz*	
□	1	0	0	0	0	0	1	0	0	1	1	0
⬛	0	0	1	0	0	0	1	0	0	1	0	1
■	0	0	1	0	0	0	1	0	0	1	1	0
●	0	1	0	1	0	0	0	1	0	0	1	0
△	1	0	0	0	1	0	0	0	1	0	0	1
◆	0	0	1	0	0	1	0	0	0	1	1	0
○	1	0	0	1	0	0	0	1	0	0	1	0

Let $s := \{$■, ◆$\}$ and $\mathbf{A} := \{\mathtt{b}, 4, \mathtt{S}\}$.

$$
\begin{aligned}
|\mathtt{P}]\, s &= \{\mathtt{b}, \mathtt{s}, 4, \mathtt{S}\} \\
&\quad \cap \{\mathtt{b}, \mathtt{d}, 4, \mathtt{S}\} \\
&= \{\mathtt{b}, 4, \mathtt{S}\} \\
[\mathtt{P}|\, \mathbf{A} &= \{⬛, ■, ◆\} \\
&\quad \cap \{□, ⬛, ■, ◆\} \\
&\quad \cap \{□, ■, ●, ◆, ○\} \\
&= \{■, ◆\}
\end{aligned}
$$

Hence, $|\mathtt{P}]\, s = \mathbf{A}$ and $[\mathtt{P}|\, \mathbf{A} = s$.

Fig. 3. The concept $\langle s, \mathbf{A} \rangle$ of "small black quadrangles".

(2) We show that $\mathbf{e} := \mathbf{c} \sqcap \mathbf{d} \sqsubseteq \mathbf{c}, \mathbf{d}$, i.e. $\mathbf{e}$ is a lower bound. Trivially, $\mathbf{e}$'s extent is a subset of both $\mathbf{c}$'s and $\mathbf{d}$'s extents: $\mathit{ext}(\mathbf{e}) = \mathit{ext}(\mathbf{c}) \cap \mathit{ext}(\mathbf{d}) = s \cap t \subseteq s, t = \mathit{ext}(\mathbf{c}), \mathit{ext}(\mathbf{d})$. It remains to be shown that $\mathit{int}(\mathbf{c} \sqcap \mathbf{d}) \supseteq \mathbf{A}, \mathbf{B}$. From equation (12) and $\mathbf{c}, \mathbf{d}$ being preconcepts we know that

$$\mathbf{A} \subseteq |\mathtt{P}]\, [\mathtt{P}|\, \mathbf{A} \text{ and } \mathbf{B} \subseteq |\mathtt{P}]\, [\mathtt{P}|\, \mathbf{B}.$$

Since $\mathbf{A}, \mathbf{B} \subseteq \mathbf{A} \cup \mathbf{B}$ we have by antitony of $|\ ]$:

$$
\begin{aligned}
|\mathtt{P}]\, [\mathtt{P}|\, \mathbf{A} \cup \mathbf{B} &\supseteq |\mathtt{P}]\, [\mathtt{P}|\, \mathbf{A} \supseteq \mathbf{A} \\
|\mathtt{P}]\, [\mathtt{P}|\, \mathbf{A} \cup \mathbf{B} &\supseteq |\mathtt{P}]\, [\mathtt{P}|\, \mathbf{B} \supseteq \mathbf{B}.
\end{aligned}
$$

Hence, $\mathbf{c} \sqsupseteq \mathbf{c} \sqcap \mathbf{d} \sqsubseteq \mathbf{d}$.

(3) We show that $\mathbf{c} \sqcap \mathbf{d}$ is the greatest lower bound of $\mathbf{c}$ and $\mathbf{d}$. This follows directly from the fact that $\cap$ is the infimum operator on sets:

$$
\begin{tikzcd}
\langle s, \mathbf{A} \rangle \arrow[dr, dash] \arrow[ddr, dash] & & \langle t, \mathbf{B} \rangle \arrow[dl, dash] \arrow[ddl, dash] \\
& \langle u, \mathbf{C} \rangle \arrow[d, dash] & \\
& \langle s \cap t, \mathbf{D} \rangle &
\end{tikzcd}
$$

Assume $\mathbf{e} = \langle u, \mathbf{C} \rangle$ (see diagram). If $s = t$, then $s \cap t = s$ s.t. $u = t$, too (and also, $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D}$). If $u = s$, then $\mathbf{C} = \mathbf{A}$ (and similarly for $u = t$). If all s, t, u are pairwise unequal, $s \cap t$ is by definition the greates subset of both s and t; hence it must coincide with u. This implies $\mathbf{C} = \mathbf{D}$ and we are done.

(4) We show that there is a greatest element $\top = \langle \mathcal{U}, |\mathtt{P}]\, \mathcal{U} \rangle$ and a smallest element $\bot = \langle [\mathtt{P}|\, \mathbb{A}, \mathbb{A} \rangle$: Both $\bot$ and $\top$ are concepts by definition. $\mathit{ext}(\bot)$ coincides with the smallest element on $\wp(\mathfrak{U})$, $\mathit{ext}(\top)$ with the smallest element on $\wp(\mathbb{A})$. From the argument in (3) it follows that $\bot$ and $\top$ are indeed the smallest and greatest element in $\mathit{Con}(\mathfrak{K})$.

Figure 3 displays a concept for the attribute based information system defined in figure 1.

2.2 Feature based concepts

In principle there is no difference in working with features $\mathbb{F}$ or attributes $\mathbb{A}$ derived from them. The definitions of domain operations and the interpretation of sets as *attributes* as in (4) shows that operations are based on in the information whether an object has a certain property or not or whether a query has a certain result: $x\mathtt{P}a \iff \langle x,a\rangle\,\mathtt{Q}\mathbf{1}$. When using multi-valued features, a single attribute a corresponds to a certain pair f and $y \in V_f$ (see figure 1) which would result in $\neg(x\mathtt{P}a) \iff \langle x,f\rangle\,\overline{\mathtt{Q}}y$ or, equivalently, $f(x) \in V_f - \{y\}$. This is, intuitively, incompatible with the notion of $\langle\mathtt{Q}|\,\mathbf{F}$. If $x\mathtt{P}a$ for $f(x) = v_f$ and $x\mathtt{P}b$ for $f(x) = w_f$, $\langle\mathtt{P}|\,\{a,b\}$ is well defined but $\langle\mathtt{Q}|\,f$ is not for we cannot express whether $f(x) = v_f$ or $f(x) = w_f$. So, for the ease of using multi-valued features instead of attribute sets, we define parametrised versions of our domain operators: What may appear overly complicated now will make things much easier in section 6.1 when we treat sets and attributes in a unified algebraic setting.

Weak (pre-) images. We first consider the usual (weak) domain operators:

$$\begin{aligned}
|\mathtt{Q}\rangle\, s\ (V_0,\ldots,V_{n-1}) &:= \{f_i \in \mathbb{F} :\ \exists x \in s : \exists y \in V_i : \langle x,f_i\rangle\,\mathtt{Q}y\} \\
&= \bigcup_{x\in s} \{f_i \in \mathbb{F} :\ |\mathtt{Q}\rangle\,\langle x,f_i\rangle \cap V_i \neq \emptyset\} \\
\langle\mathtt{Q}|\,\mathbf{F}\ (x'_0,\ldots,x'_{m-1}) &:= \left\{x \in \mathcal{U} :\ \forall f \in \mathbf{F} : \exists j \in \mathbf{m} : f(x'_j) = f(x)\right\} \\
&= \bigcap_{i\in\mathbf{n}} \bigcup_{j\in\mathbf{m}} \left\{x \in \mathcal{U} :\ f_i(x) = f_i(x'_j)\right\}
\end{aligned} \tag{31}$$

for $V_i \subseteq \operatorname{codom}(f_i)$ and $s' = \left\{x'_j : j \in \mathbf{m}\right\} \subseteq \mathcal{U}$. The additional parameters $(V_i)_{i\in\mathbf{n}}$ and $(x'_j)_{j\in\mathbf{m}}$ are to determine the "range" of the domain operators over sets of attributes in terms of admissible feature values such that

- $|\mathtt{Q}\rangle\, s(V_i)_{i\in\mathbf{n}}$ is the set of all those features for which *at least one element* $x \in s$ takes one of the admitted values in V_i under f_i and
- $\langle\mathtt{Q}|\,\mathbf{F}(x'_j)_{j\in\mathbf{m}}$ is the set of all those objects, which *for every feature* $f \in \mathbf{F}$ are indiscernible from at least one element $x \in s'$.

For example, we have with $s = \{\blacksquare, \blacksquare, \triangle\}$ and $(V_i) = (\{\mathtt{b}\}, \{\mathtt{c},\mathtt{t},\mathtt{d}\}, \{1\}, \mathtt{S})$:

$$|\mathtt{Q}\rangle\,\{\blacksquare, \blacksquare, \triangle\}\,(\{\mathtt{b}\}, \{\mathtt{c},\mathtt{t},\mathtt{d}\}, \{1\}, \mathtt{S}) = \{\mathit{col}, \mathit{shp}, \mathit{siz}\} \tag{32}$$

because $\mathit{col}(\blacksquare) = \mathtt{b} \in \{\mathtt{b}\}$, $\mathit{shp}(\triangle) = \mathtt{t} \in \{\mathtt{c},\mathtt{t},\mathtt{d}\}$, $\mathit{siz}(\blacksquare) = \mathtt{S} \in \{\mathtt{S},\mathtt{B}\}$, but $\mathit{edg}(\blacksquare) = \mathit{edg}(\blacksquare) = 4 \notin \{1\}$ and $\mathit{edg}(\triangle) = 3 \notin \{1\}$. Tracing the image set back to $\mathtt{Q}$'s domain, $\langle\mathtt{Q}|\,|\mathtt{Q}\rangle\, s$ yields a definition that is equivalent to defining

Let $s = \{\blacksquare, {\scriptstyle\blacksquare}, \triangle\}$ and $[V_i] = [\{\mathtt{b}\}, \{\mathtt{c}, \mathtt{t}, \mathtt{d}\}, \{1\}, \{\mathtt{B}\}]$.

(1) Compute $\mathbf{F}$ by $|\mathtt{Q}\rangle\, s[V_i]$:

	$\mathbf{F}$			
	col	*shp*	*edg*	*siz*
$\square$				
$\blacksquare$	b	s	4	B
${\scriptstyle\blacksquare}$	b	s	4	S
$\bullet$				
$\triangle$	w	t	3	B
$\blacklozenge$				
$\bigcirc$				

$\mathbf{F} = \{col, shp, siz\}$

(2) Compute $\langle\mathtt{Q}|\, \mathbf{F}(s)$:

	$\|f\rangle\, s$		
	col	*shp*	*siz*
$\square$	w	s	S
$\blacksquare$	b	s	B
${\scriptstyle\blacksquare}$	b	s	S
$\bullet$	d	c	S
$\triangle$	w	t	B
$\blacklozenge$	b	d	S
$\bigcirc$	w	c	S

$\langle\mathbf{F}\rangle s = \{\square, \blacksquare, {\scriptstyle\blacksquare}, \triangle\}$.

Fig. 4. s-similar objects: $\langle\mathtt{Q}|\,(|\mathtt{Q}\rangle\, s[V_i])(s) = \langle\mathbf{F}\rangle s$

membership by a simple first order clause in *conjunctive normal form*:

$$
\begin{aligned}
& x \in \langle\mathtt{Q}|\,(|\mathtt{Q}\rangle\,(\{\blacksquare, {\scriptstyle\blacksquare}, \triangle\})\,(\{\mathtt{b}\}, \{\mathtt{c}, \mathtt{t}, \mathtt{d}\}, \{1\}, \mathtt{S}))\,(\blacksquare, {\scriptstyle\blacksquare}, \triangle) \\
\overset{(32)}{\iff}\; & x \in \langle\mathtt{Q}|\,\{col, shp, siz\}\,(\blacksquare, {\scriptstyle\blacksquare}, \triangle) \qquad (33) \\
\iff\; & \quad col(x) = \mathtt{b} \lor col(x) = \mathtt{w} \\
& \land\, shp(x) = \mathtt{t} \lor shp(x) = \mathtt{s} \\
& \land\, siz(x) = \mathtt{S} \lor siz(x) = \mathtt{B} \\
\iff\; & x \in \quad (\{\blacksquare, {\scriptstyle\blacksquare}, \blacklozenge\} \cup \{\square, \triangle, \bigcirc\}) \\
& \qquad \cap (\{\square, \blacksquare, {\scriptstyle\blacksquare}\} \cup \{\triangle\}) \\
& \qquad \cap (\{\square, {\scriptstyle\blacksquare}, \blacklozenge, \bigcirc\} \cup \{\blacksquare, \triangle\}) \\
\iff\; & x \in \{\square, \blacksquare, {\scriptstyle\blacksquare}, \triangle\}
\end{aligned}
$$

It is the set of all objects x for which all elements in $\{\blacksquare, {\scriptstyle\blacksquare}, \triangle\}$ are indistinguishable from x — supposing that we are unable to tell black from white, squares from triangles and Small ones from Big ones. $\bullet$ *is* distinguishable, because it is dark and $\blacklozenge$ and $\bigcirc$ are neither squares nor triangles. The calculation is shown in figure 4. Equation (32) describes supersets of "similar" objects where similarity is determined in terms of (V_i).

Weakening the requirement of similarity as defined by $V_i \subseteq \mathrm{dom}\,(f_i)$ to $V_i = \mathrm{dom}\,(f_i)$ and by assuming $\mathtt{Q}$ to be clear from the formal context we write

$$\langle\mathbf{F}\rangle s := \langle\mathtt{Q}|\,(|\mathtt{Q}\rangle\, s(\mathrm{dom}\,(f_i))_{i\in\mathbf{n}})(s).$$

(the expression $\langle s\rangle\mathbf{F}$ ist defined by exchange of (pre-) image operators). For reasons that will become clear later (sections 6.1-6.2), we also note $s \subseteq \langle\mathbf{F}\rangle s$.[6]

[6] The proposition follows directly from the definition: $\langle\mathtt{Q}|\,\mathbf{F}(s) \subseteq \mathcal{U}$ since $x \in \mathcal{U} \supseteq s$ and every object $x \in s$ is indiscernible from itself such that $s \subseteq \langle\mathtt{Q}|\,\mathbf{F}(s) \subseteq \mathcal{U}$.

Let $s = \{□, ■, ▪, ●, ◆\}$ and $(V_i)_{i \in \mathbf{m}} = (\{\mathtt{b}, \mathtt{w}\}, \{\mathtt{c}, \mathtt{t}\}, \{3\}, \{\})$.

(1) Compute $\mathbf{F}$ by $|\mathtt{Q}]\, s(V_i)_{i \in \mathbf{m}}$:

	F			
	col	*shp*	*edg*	*siz*
□	w	s	4	S
■	b	s	4	B
▪	b	s	4	S
●	d	c	1	S
△				
◆	b	d	4	S
○				

$\mathbf{F} = \{col, shp\}$

(2) Compute $[\mathtt{Q}|\, \mathbf{F}(s)$:

f	$\lvert f]\, s$ *col*	*shp*
□	w	s
■	b	s
▪	b	s
●	d	c
△	w	t
◆	b	d
○	w	c

$[\mathbf{F}]\, s = \{■, ▪, ◆\}$.

Note that $V_3 = \emptyset$ results in "ignorance" of $f_3 = siz$.

Fig. 5. Identifiable objects in s: $[\mathtt{Q}|\, (|\mathtt{Q}]\, s(V_i))(s) = [\mathbf{F}]\, s$

Strict (pre-) images. An according definition of strict domain operators is

$$
\begin{aligned}
|\mathtt{Q}]\, s\, (V_0, \ldots, V_{n-1}) &:= \{f_i \in \mathbb{F} :\ \forall x \in s : \exists y \in V_i : \langle x, f_i \rangle\, \mathtt{Q} y\} \\
&= \bigcap_{x \in s} \{f_i \in \mathbb{F} :\ |\mathtt{Q}\rangle\, \langle x, f_i \rangle \subseteq V_i\}
\end{aligned} \tag{34}
$$

We now need to abbreviate $s := \{x_i : i \in \mathbf{m}\}$. Then,

$$
\begin{aligned}
[\mathtt{Q}|\, \mathbf{F}(x'_0, \ldots, x'_{m-1}) &= [\mathtt{Q}|\, \mathbf{F}(s) \qquad (35) \\
&:= \left\{ x \in \mathcal{U} :\ \begin{array}{c} \forall f \in \mathbf{F} : \forall x' \in s : \\ f(x') = f(x) \longrightarrow x \in s \end{array} \right\} \\
&= \bigcap_{i \in \mathbf{m}} \bigcup_{y \in V_i} \{x \in s :\ x \in \langle f|\, y \longrightarrow \langle f|\, y \subseteq s\}
\end{aligned}
$$

Note that even though f is a feature and any x takes exactly one value under f we still need to examine all $y \in V_i$ for it could be that s includes several such classes (see example in figure 5; it includes all gray and black objects).

Similar to the simplification of the weak (pre-) image operator above, we also define

$$
[\mathbf{F}]\, s := [\mathtt{Q}|\, \mathbf{F}(s) = [\mathtt{Q}|\, (|\mathtt{Q}]\, s(\mathrm{dom}\,(f_i))_{i \in \mathbf{n}})(s). \tag{36}
$$

Also, we note again that $[\mathbf{F}|\, s \subseteq s$ (and again point out its relevance for 6.1-6.2). In the following, we will establish a connection between $\langle\ \rangle$ and $[\ ]$ and upper and lower approximations from rough set theory. We will, in contrast to standard literature on data analysis by formal concept analysis or rough sets, use an algebraic approach. This, finally, allows us to prove several theorems from FCA in a more general and simpler setting; and it also establishes a connection to modal logics as it was intended by the box- and diamond like notation of (weak) images.

3 Rough set data analysis: objects and definability

Rough set analysis, [17, 18], explores whether one can define subsets $s \subseteq \mathcal{U}$ using knowledge encoded by equivalence relations rather than attributes. For some feature f we define $\widetilde{f}$ to be the equivalence relation induced by f: $x\widetilde{f}y :\Longleftrightarrow f(x) = f(y)$. Equivalently, $x\widetilde{f}y \Longleftrightarrow |\mathtt{Q}\rangle\langle x, f\rangle = |\mathtt{Q}\rangle\langle y, f\rangle$. This implies $|\mathtt{Q}]\langle x, f\rangle = |\mathtt{Q}]\langle y, f\rangle$ and, hence,

$$\begin{aligned} x\widetilde{f}y \Longleftrightarrow \quad & [\mathtt{Q}|\,(|\mathtt{Q}]\,\{x\}[f(x)])(\{x\}) \\ &= [\mathtt{Q}|\,\{f\}(\{x\}) \\ &= [\{f\}]\,\{x\} = [\{f\}]\,\{y\} \end{aligned}$$

and, for $\{x\}$ and $\{f\}$ are singletons, the same for weak domain operators:

$$x\widetilde{f}y \Longleftrightarrow \langle\{f\}\rangle\,\{x\} = \langle\{f\}\rangle\,\{y\}\,.$$

For the ease of writing, we shall write $\mathbb{R}$ for $\left\{\widetilde{f} : f \in \mathbb{F}\right\}$ (or $\mathbb{A}$ instead of $\mathbb{F}$) and similarly $\mathbf{R} \subseteq \mathbb{R}$. When clear from context, we use $R, P, Q \in \mathbb{R}$ as symbols for equivalences $\widetilde{f}, \widetilde{g}, \widetilde{h}$ and $f, g, h \in \mathbb{F}$.

3.1 Upper and lower approximations

Since equivalences are closed under intersection, we define the *indiscernability relation (induced by* $\mathbf{R}$*)* by

$$\widetilde{\widetilde{\mathbf{R}}} := \bigcap_{R\in\mathbf{R}} R.^{7} \tag{37}$$

The knowledge of $\mathfrak{I}$ is the set of all formulae that can be built from all subsets of $\mathbb{R}$ and its indiscernability relations:

$$\mathrm{KB}(\mathfrak{I}) := \bigcup_{\mathbf{R}\subseteq\mathbb{R}} \left\{\widetilde{\widetilde{\mathbf{R}}}\right\}. \tag{38}$$

Note that by distributivity and the definition of $\approx$ as $\bigcap$, the above expression can be transformed into a conjunctive normal form similar to the one in equation (32). Trivially, $\widetilde{\widetilde{R}} \subseteq R$ for any $R \in \mathbf{R}$ and, furthermore, $[x]_{\widetilde{\widetilde{\mathbf{R}}}} \subseteq [x]_R$ for any

[7] The name "indiscernability" simply reflects the fact that two elements cannot be distinguished because they have exactly the same properties: $x\widetilde{\widetilde{\mathbf{R}}}y \Longleftrightarrow \forall R \in \mathbf{R} : xRy \Longleftrightarrow \forall f \in \mathbf{F} : f(x) = f(y)$. Note that any $R \in \mathrm{EquR}(s)$, R can be interpreted as some $\widetilde{\widetilde{\mathbf{R}}}$; in the trivial case it is $R = \widetilde{\widetilde{\mathbf{R}}}$ for $\mathbf{R} = \{R\}$.

$x \in \mathcal{U}$ and $R \in \mathbf{R}$, too. KB($\mathfrak{I}$) defines a complete lattice with a special supremum operator:

$$P \sqcup Q := (P \cup Q)^* \tag{39}$$

and $\bigsqcup \mathbb{R} \subseteq \mathbb{T}$ as greatest and $\bigcap \mathbb{R} \supseteq \mathit{1}_{\mathcal{U}}$ as smallest element. Trivially, $\wp(\mathcal{U})$ is the canonical powerset lattice. We give two proofs: The first one is a pointwise proof based on the definition of $\bigsqcup$ as given in [4] and is similar to the one presented in [1]. The second proof only takes three lines by using the equivalent definition in equation (39) above:

Proof (Equation 39). We first show that, $P \sqcup Q$ is an equivalence given that P and Q are equivalences.
Reflexivity. Instead of breaking down the equation and reducing the claim to showing that P and Q are reflexive, one quickly observes the validity by the inclusion of $\mathit{1}$ in the reflexive transitive closure *.
Transitivity. [4] give an equivalent pointwise definition of $\sqcup$ which requires a lengthy proof:

$$\begin{aligned}
& x(P \sqcup Q)y \wedge y(P \sqcup Q)z \\
\iff & \exists n \in \mathbb{N}_0 \exists R_0, \dots, R_{i-1} \in \{P, Q\} : x\, R_0 ; R_1 ; \cdots ; R_{n-1}\; y \\
& \wedge \exists m \in \mathbb{N}_0 \exists R'_0, \dots, R'_{m-1} \in \{P, Q\} : y\, R'_0 ; R'_1 ; \cdots ; R'_{m-1}\; z \\
\iff & \exists n, m \in \mathbb{N}_0 \exists R_0, \dots, R_{n-1}, R'_0, \dots, R'_{m-1} \in \{P, Q\} : \\
& x\, R_0 ; R_1 ; \cdots ; R_{n-1}\; y \wedge y\, R'_0 ; R'_1 ; \cdots ; R'_{m-1}\; z \\
\implies & \exists k \in \mathbb{N}_0 \exists R_0, \dots, R_{k-1} \in \{P, Q\} : x\, R_0 ; R_1 ; \cdots ; R_{k-1}\; z \\
\iff & x(P \sqcup Q)z
\end{aligned}$$

for some $k \leq m + n$. Algebraically, we derive

$$(P \sqcup Q);(P \sqcup Q) \overset{39}{=\!=} (P \cup Q)^*;(P \cup Q)^* \overset{*}{=\!=} (P \cup Q)^* \overset{39}{=\!=} P \sqcup Q. \tag{40}$$

Symmetry. Pointwise, this again requires a lengthy derivation:

$$\begin{aligned}
& x(P \sqcup R)y \\
\iff & \exists n \in \mathbb{N}_0 \exists Q_0, \dots, Q_{n-1} \in \{P, R\} : x\, Q_0 ; Q_1 ; \cdots ; Q_{n-1}\; y \\
& |:\; P, Q \in \text{EquR},\; P^{\smile} ; R^{\smile} = (\, P ; R\,)^{\smile} \text{ and associativity} \\
\implies & \exists n \in \mathbb{N}_0 \exists Q_0, \dots, Q_{n-1} \in \{P, R\} : y\, Q^{\smile}{}_{n-1} ; Q^{\smile}{}_{n-2} ; \cdots ; Q^{\smile}{}_0\; x \\
\iff & \exists n \in \mathbb{N}_0 \exists Q_0, \dots, Q_{n-1} \in \left\{P^{\smile}, R^{\smile}\right\} : y\, Q_{n-1} ; Q_{n-2} ; \cdots ; Q_0\; x \\
& |:\; P, Q \in \text{EquR},\; P^{\smile} ; R^{\smile} = (\, P ; R\,)^{\smile} \text{ and associativity} \\
\implies & \exists n \in \mathbb{N}_0 \exists Q_0, \dots, Q_{n-1} \in \left\{P^{\smile}, R^{\smile}\right\} : x(\, Q_0 ; Q_1 ; \cdots ; Q_{n-1}\;)^{\smile} y \\
& |:\; P, Q \text{ symmetric.} \\
\implies & \exists n \in \mathbb{N}_0 \exists Q_0, \dots, Q_{n-1} \in \{P, R\} : y(\, Q_0 ; Q_1 ; \cdots ; Q_{n-1}\;)x \\
\iff & y(P \sqcup R)x
\end{aligned}$$

Algebraically, symmetry simply follows from symmetry of $\cup$:

$$P \sqcup Q \stackrel{39}{=\!=} (P \cup Q)^* \stackrel{\cup}{=\!=} (Q \cup P)^* \stackrel{39}{=\!=} Q \sqcup P \tag{41}$$

and we are done. *Remark.* Of course $P \sqcup P = P$, since we then have sequences of compositions of $Q \in \{P\}$, i.e. $Q = P$ and $P \fatsemi P \subseteq P$. Also, $P \sqcup R = R \sqcup P$ since $Q \in \{P, R\} = \{R, P\}$. We conclude that $P \sqcup R$ is indeed an equivalence. Finally, we need to show that

$\sqcup$ *is a least upper bound.* Again, an algebraic argument is much simpler: Assume that $P, Q \subset R \subset P \sqcup Q = (P \cup Q)^*$ where P, Q and R are equivalences. The contradiction is obvious, for the *-closure is the *smallest* such element. In detail, since R is assumed to be an equivalence, $P \cup Q \subset R = R^* \subset (P \cup Q)^*$. By isotony and idempotence of *, we then have

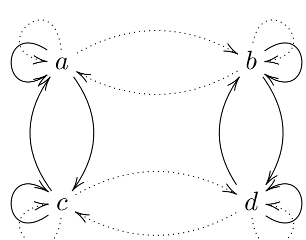
$$(P \cup Q)^* \subset R = R^* = R^{*^*} \subset (P \cup Q)^{*^*} = (P \cup Q)^*$$

which is a contradiction. $\square$

The fact that

$$x(P \sqcup R)y \iff \exists n \in \mathbb{N}_0 \exists Q_0, \ldots, Q_{n-1} \in \{P, R\} : x\, Q_0 \fatsemi \cdots \fatsemi Q_{n-1}$$

can be motivated intuitively by two graph relations P and R on the same set $\mathcal{U}$ of vertices. With P, R being equivalence relations, both relations partition the entire graph into connected components. Then, x relates to y via $P \sqcup R$, if both belong to one and the same P- or R-class or if we find a sequence of pairs of nodes that allow for an alternating sequence of P- and R-"jumps" within and between the corresponding equivalence classes. For example, with $P = \longrightarrow$ and $R = \dashrightarrow$ in



we then have

$$dPbRaPc \text{ or } dRc \text{ or } dRcPaRaPc.$$

We start with an arbitrary sequence:

$$
\begin{aligned}
& x = x_0 Q_0 x_1 \cdots x_{n-2} Q_{n-1} x_{n-1} = y \\
& |: \ Q_i \in \{P, R\} \\
\Longrightarrow\ & x = x_0 (P \cup R) x_1 \cdots x_{n-2} (P \cup R) x_{n-1} = y \\
& |: \ P, R \in \mathrm{EquR} \Longrightarrow (P \cup R) \subseteq (P \cup R)^* \\
\Longrightarrow\ & x = x_0 (P \cup R)^* x_1 \cdots x_{n-2} (P \cup R)^* x_{n-1} = y \\
& |: \ (P \cup R)^* = \ P^* ⨾ (\ R ⨾ P^* \ )^* \\
\Longleftrightarrow\ & x = x_0 \ P^* ⨾ (\ R ⨾ P^* \ )^* \ x_1 \cdots x_{n-2} \ P^* ⨾ (\ R ⨾ P^* \ )^* \ x_{n-1} = y \\
& |: \ P, R \in \mathrm{EquR} \\
\Longleftrightarrow\ & x = x_0 \ P ⨾ (\ R ⨾ P \ )^* \ x_1 \cdots x_{n-2} \ P ⨾ (\ R ⨾ P \ )^* \ x_{n-1} = y \\
& |: \ \text{pointwise splitting of composition} \\
\Longleftrightarrow\ & x = x_0 P y_0 (\ R ⨾ P \ )^* x_1 \cdots x_{n-2} P y_{n-2} (\ R ⨾ P \ )^* x_{n-1} = y \\
& |: \ \ R ⨾ P ⨾ R \ = \ R ⨾ (\ P ⨾ R \ )^* \\
\Longleftrightarrow\ & x = x_0 \ P ⨾ \ R ⨾ P \ x_1 \cdots x_{n-2} \ P ⨾ \ R ⨾ P \ x_{n-1} = y \\
& |: \ \text{Symmetry of } \cup \text{ and line (17)} \\
\Longleftrightarrow\ & x = x_0 \ R ⨾ \ P ⨾ R \ x_1 \cdots x_{n-2} \ R ⨾ \ P ⨾ R \ x_{n-1} = y
\end{aligned}
$$

The *lower R-approximation of s* is the union set of all R–equivalence classes that are contained in s, and the corresponding *upper approximation* is the union of all classes containing at least one element of s:

$$[\![R]\!]s := \{x \in \mathcal{U} : [x]_R \subseteq s\} \tag{42}$$

$$\langle\!|R|\!\rangle s := \{x \in \mathcal{U} : [x]_R \cap s \neq \emptyset\}\,. \tag{43}$$

Clearly, $[\![R]\!]s \subseteq s \subseteq \langle\!|R|\!\rangle s$. Also, the pointwise definition of $\langle\!| \ |\!\rangle$ and $[\![\ ]\!]$ in equation (42) reflects the weak and strict domain operators $| \, \rangle$ and $| \,]$ as in equations (31) and (34-35). For $\mathbf{R} = \left\{ \widetilde{col}, \widetilde{shp}, \widetilde{edg} \right\}$ and $s = \{■, ●, △\}$,

$$\{●, △\} = [\![\mathbf{R}]\!]\,\{■, ●, △\} \ \subset \ \langle\!|\mathbf{R}|\!\rangle\,\{■, ●, △\} = \{█, ■, ●, △\}\,. \tag{44}$$

A set $s \subseteq \mathcal{U}$ is *roughly R-definable*, if $[\![R]\!]s \neq \emptyset$ or $\langle\!|R|\!\rangle s \neq \mathcal{U}$. A set s is called *(exactly) R-definable*, if

$$[\![R]\!]s = s \Longleftrightarrow \langle\!|R|\!\rangle s = s. \tag{45}$$

Upper and lower approximations are dual operations in the usual sense of modal logics:

$$[\![\mathbf{R}]\!]\overline{s} = \overline{\langle\!|\mathbf{R}|\!\rangle s}. \tag{46}$$

The proof is deferred to section 5.

Interpreting $\langle\!|R|\!\rangle s - [\![R]\!]s$ as a region of vagueness, we have a three-valued characteristic function for a three-valued logic (Ł$_3$). One can also define four different membership relations relative to R:

$$\begin{array}{rlrl} x \sqsubseteq_R s :\Longleftrightarrow & x \in [\![R]\!]s & x \lessdot_R s :\Longleftrightarrow & x \in \langle\!|R|\!\rangle s \\ \neg x \sqsubseteq_R s :\Longleftrightarrow & x \notin \langle\!|R|\!\rangle s & \neg x \lessdot_R s :\Longleftrightarrow & x \notin [\![R]\!]s \\ \Longleftrightarrow & x \in [\![R]\!]\overline{s} & \Longleftrightarrow & x \in \langle\!|R|\!\rangle \overline{s} \end{array} \tag{47}$$

which leaves us with possible interpretations in intuitionistic or paraconsistent logics, [5, 16, 13].
Of course, we can choose $s = \mathcal{U}$ or $s = \emptyset$ and find that

$$[\![\mathbf{R}]\!]\mathcal{U} = \mathcal{U} = \langle\!|\mathbf{R}|\!\rangle\mathcal{U} \text{ and } [\![\mathbf{R}]\!]\emptyset = \emptyset = \langle\!|\mathbf{R}|\!\rangle\emptyset \tag{48}$$

for any set $\mathbf{R}$ of equivalences on $\mathcal{U}$. Furthermore, any equivalence class $[x]_{\approx\atop\mathbf{R}}$ is invariant under $\mathbf{R}$-approximations:

$$[\![\mathbf{R}]\!][x]_{\approx\atop\mathbf{R}} = [x]_{\approx\atop\mathbf{R}} = \langle\!|\mathbf{R}|\!\rangle[x]_{\approx\atop\mathbf{R}}. \tag{49}$$

It also seems useful to lift approximation operators to classifications or quotients.

A classification is a family of subsets $\{c_i \subseteq \mathcal{U} : i \in \mathbf{k}\}$ of the base set $\mathcal{U}$; usually, one assumes pairwise disjointness and $\bigcup c_i = \mathcal{U}$. Then, classifications are partitions that correspond to a quotient induced by an equivalence. Given a classification $\mathfrak{c}$, we can define $f : \mathcal{U} \to \mathbf{k}$ with $f(x) = i :\Longleftrightarrow x \in c_i$ such that $\mathfrak{c} = \mathcal{U}/\widetilde{f}$.[8] The usual definition of approximations of classifications is

$$[\![\mathbf{R}]\!]\mathfrak{c} = \{[\![\mathbf{R}]\!]c_i : c_i \in \mathfrak{c}\} \text{ and } \langle\!|\mathbf{R}|\!\rangle\mathfrak{c} = \{\langle\!|\mathbf{R}|\!\rangle c_i : c_i \in \mathfrak{c}\}. \tag{50}$$

Assuming $\widetilde{\widetilde{\mathbf{R}}} = \widetilde{f}$ and $\mathfrak{c} = U/\widetilde{g}$,

$$[\![\mathbf{R}]\!]\mathfrak{c} = \bigcup_{v \in \mathrm{dom}(g)} \{[\![\mathbf{R}]\!] \langle g|\, v\} = \bigcup_{v \in \mathrm{dom}(g)} \left\{\left\{x : |\widetilde{f}\rangle\, x \subseteq \langle g|\, v\right\}\right\}. \tag{51}$$

Since for every x in $[\![\mathbf{R}]\!] \langle g|\, v$ we trivially have $|g\rangle\, x = \{v\}$ such that

$$\begin{aligned} x \in [\![\mathbf{R}]\!]s &\Longleftrightarrow x \in [\![\widetilde{f}]\!]\langle g|\, v \\ &\Longleftrightarrow x \in [\![\widetilde{f}]\!]\langle g|\, |g\rangle\, x \\ &\Longleftrightarrow |\widetilde{f}\rangle\, \langle g|\, |g\rangle\, x \subseteq \langle g|\, |g\rangle\, x. \end{aligned} \tag{52}$$

The simple case of $[\![\mathbf{R}]\!]s$ can be seen as a special case where $\mathfrak{c} = \{s\}$ is an abbreviation for $\{s, \overline{s}\}$ such that equation (50) actually implies equations (42,43) as a special case. This classification is induced by the characteristic function (i.e. attribute) $\dot{s} : \mathcal{U} \to \mathbf{2}$.

[8] For the sake of readability we will not distinguish between the approximations of sets, classifications, quotients or functions when clear from context: $\mathcal{U}/\widetilde{f} = \mathcal{U}/f$ and $[\![R]\!]\mathfrak{c} = [\![R]\!]f$.

The comparison of two different classifications

$$\mathfrak{c} = \{\{\blacksquare, \blacksquare, \blacklozenge, \bullet\}, \{\square, \triangle, \bigcirc\}\} \text{ and } \mathfrak{d} = \{\{\square, \blacksquare, \blacksquare\}, \{\triangle, \blacklozenge\}, \{\bullet, \bigcirc\}\} \tag{53}$$

gives rise several questions: Can we use $\mathfrak{c} = \mathcal{U}/Q$ to describe $\mathfrak{d} = \mathcal{U}/P$ (or vice versa)? Can we compare Q to P? And, finally, given some $\mathfrak{c}$ and two sets of equivalences $\mathbf{P}$ and $\mathbf{R}$ can we compare $\mathbf{P}$ and $\mathbf{R}$ w.r.t Q?

3.2 Utility of knowledge

The interesting thing is to compare the descriptive power of equivalences $\mathbf{P}$ to another set of equivalences $\mathbf{Q}$.
Pointwise speaking, $\mathbf{P}$ is more informative than $\mathbf{Q}$, if it is able to correctly classify more objects against the reference classification: We define the $\mathbf{P}$*-positive set of* $\mathbf{Q}$ *against* $\mathfrak{c}$ to be the union set of all $\mathbf{P}$-lower approximations of $\overset{\approx}{\widetilde{\mathbf{Q}}}$ classes on the reference classification. Recall that

$$\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket s := \bigcup_{c \in s/\overset{\approx}{\widetilde{\mathbf{Q}}}} \llbracket \mathbf{P} \rrbracket c. \tag{54}$$

Again, we can lift the definition to $\mathfrak{c} = \mathcal{U}/Q$:

$$\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \mathfrak{c} \ := \bigcup_{c \in \mathfrak{c}} \llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket c \ = \bigcup_{c \in \mathcal{U}/Q} \ \bigcup_{s \in c/\overset{\approx}{\widetilde{\mathbf{Q}}}} \llbracket \mathbf{P} \rrbracket s \, . \tag{55}$$

Note that $\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \mathfrak{c} \subseteq \mathcal{U}$ is a *flat* set and not a set of class approximations like $\llbracket R \rrbracket \mathfrak{c}$. It is a simple collection of objects in $\mathcal{U}$ for which $\mathbf{P}$-knowledge suffices to describe $\mathbf{Q}$-knowledge on *any* Q-class. Since Q-classes are induced by some $f \in \mathbb{F}$, the information $f(x)$ is lost for $x \in \llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \mathcal{U}/\widetilde{f}$. Should we be interested in just a simple set $s \subseteq \mathcal{U}$, we define $\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket s := (\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \{s, \overline{s}\}) \cap s$. If the $\mathbf{P}$-positive set of $\mathbf{R}$ includes the $\mathbf{Q}$-positive set of $\mathbf{R}$ (with respect to a reference partition $\mathfrak{c} = \mathcal{U}/Q$), $\mathbf{P}$ obviously contains more $\mathbf{R}$-knowledge than $\mathbf{Q}$. We then write

$$\mathbf{P} \overset{\mathbf{R}}{\succeq}_{\mathfrak{c}} \mathbf{Q} \ :\Longleftrightarrow \ \llbracket \mathbf{P} \lessdot \mathbf{R} \rrbracket \mathfrak{c} \supseteq \llbracket \mathbf{Q} \lessdot \mathbf{R} \rrbracket \mathfrak{c}. \tag{56}$$

Again, we may simplify

$$\mathbf{P} \overset{\mathbf{R}}{\succeq}_{s} \mathbf{Q} := \mathbf{P} \overset{\mathbf{R}}{\succeq}_{\{s,\overline{s}\}} \mathbf{Q} \text{ and } \mathbf{P} \overset{\mathbf{R}}{\succeq} \mathbf{Q} := \mathbf{P} \overset{\mathbf{R}}{\succeq}_{\mathcal{U}} \mathbf{Q}. \tag{57}$$

Proof (Equation 57). $\mathbf{P} \overset{\mathbf{R}}{\succeq}_{s} \mathbf{Q}$ becomes $\llbracket \mathbf{P} \lessdot \mathbf{R} \rrbracket s \supseteq \llbracket \mathbf{Q} \lessdot \mathbf{R} \rrbracket s$, which by equation (55) equals $\llbracket \mathbf{P} \lessdot \mathbf{R} \rrbracket \{s, \overline{s}\} \cap s \supseteq \llbracket \mathbf{Q} \lessdot \mathbf{R} \rrbracket \{s, \overline{s}\} \cap s$. By isotony, it then holds that $\llbracket \mathbf{P} \lessdot \mathbf{R} \rrbracket \{s, \overline{s}\} \supseteq \llbracket \mathbf{Q} \lessdot \mathbf{R} \rrbracket \{s, \overline{s}\}$. For $s = \mathcal{U}$, we need to consider $\llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \{\mathcal{U}, \emptyset\}$ which, again by equation (55), becomes

$$\begin{aligned} \llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \{\mathcal{U}, \emptyset\} &= \llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \mathcal{U} \cup \llbracket \mathbf{P} \lessdot \mathbf{Q} \rrbracket \emptyset \\ &= \bigcup_{c \in \mathcal{U}/\overset{\approx}{\widetilde{\mathbf{Q}}}} \llbracket \mathbf{P} \rrbracket c \cup \bigcup_{c \in \emptyset/\overset{\approx}{\widetilde{\mathbf{Q}}}} \llbracket \mathbf{P} \rrbracket c \\ &= \llbracket \mathbf{P} \rrbracket \mathcal{U}/\overset{\approx}{\widetilde{\mathbf{Q}}} = \llbracket \mathbf{P} \rrbracket \mathfrak{c}. \square \end{aligned}$$

In most cases, properties of relation sets are compared with respect to the entire set of objects in the universe such that $Q = \top$. We then simply drop the arguments and say that $\mathbf{P} \overset{\mathbf{R}}{\succeq} \mathbf{Q}$ iff $[\![\mathbf{P} \lessdot \mathbf{R}]\!]\mathcal{U} \supseteq [\![\mathbf{Q} \lessdot \mathbf{R}]\!]\mathcal{U}$.[9] If $\mathcal{U}/Q = \{s, \overline{s}\}$ the following equivalence shows that "more knowledge" as expressed by $\succeq$ simply means bigger regions of lower approximations for both s and its complement:

$$\mathbf{P} \overset{Q}{\succeq} \mathbf{R} \iff [\![\mathbf{P}]\!]s \cup \overline{\langle\!|\mathbf{P}|\!\rangle s} \supseteq [\![\mathbf{R}]\!]s \cup \overline{\langle\!|\mathbf{R}|\!\rangle s}. \tag{58}$$

Roughly speaking, the attributes $\mathbf{P}$ are better suited to describe the attribute Q than the attributes $\mathbf{R}$.

Proof (Equation 58). Pointwise proof by simple replacement of term definitions:

$$\begin{aligned}
\mathbf{P} \overset{Q}{\succeq} \mathbf{R} &\iff \mathbf{P} \overset{Q}{\succeq}_{\mathcal{U}} \mathbf{R} \\
&\qquad |\text{: Defn. } \succeq \\
&\iff [\![\mathbf{P} \lessdot Q]\!]\mathcal{U} \supseteq [\![\mathbf{R} \lessdot Q]\!]\mathcal{U} \\
&\qquad |\text{: Defn. } \lessdot \\
&\iff \bigcup_{c \in \mathcal{U}/Q} [\![\mathbf{P}]\!]c \supseteq \bigcup_{c \in \mathcal{U}/Q} [\![\mathbf{R}]\!]c \\
&\qquad |\text{: Disjointness of equivalence classes and lower approx.} \\
&\iff [\![\mathbf{P}]\!]s \cup [\![\mathbf{P}]\!]\overline{s} \supseteq [\![\mathbf{R}]\!]s \cup [\![\mathbf{R}]\!]\overline{s} \\
&\qquad |\text{: Duality of } [\![\ ]\!] \text{ and } \langle\!|\ |\!\rangle \\
&\iff [\![\mathbf{P}]\!]s \cup \overline{\langle\!|\mathbf{P}|\!\rangle s} \supseteq [\![\mathbf{R}]\!]s \cup \overline{\langle\!|\mathbf{R}|\!\rangle s}. \square
\end{aligned}$$

We call a relation R

- *(Q-) dispensable in* $\mathbf{R}$, if it carries only redundant information, i.e. if $[\![\mathbf{R} - \{R\}]\!]\mathfrak{c} = [\![\mathbf{R}]\!]\mathfrak{c}$ and $\langle\!|\mathbf{R} - \{R\}|\!\rangle\mathfrak{c} = \langle\!|\mathbf{R}|\!\rangle\mathfrak{c}$ [10] and
- *indispensable* otherwise.
- A set $\mathbf{R}$ is *(Q-) irreducible*, iff it does not contain any Q-dispensable relation and it is called
- *redundant* otherwise.

Removing redundant information yields "minimal" knowledge sets: We call

- $\mathbf{P} \subseteq \mathbb{R}$ a *(Q-) reduct of* $\mathbf{R} \subseteq \mathbb{R}$, iff $\mathbf{P} \subseteq \mathbf{R}$ and $\mathbf{P}$ is Q-irreducible. $\mathrm{Red}_Q(\mathbf{R})$ denotes the set of all Q-reducts of $\mathbf{R}$.
- Since reducts contain only *indispensable* relations, we define the intersection of all reducts to be the *core* $\mathrm{Cor}_Q(\mathbf{R}) := \bigcap \mathrm{Red}_Q(\mathbf{R})$. [11]

[9] Again, we have to admit another notational insufficiency: $\mathcal{U}/\top = \{\mathcal{U}\} \neq \mathcal{U}$, but we shall treat them as if they were equal.

[10] "*s-dispensability* of R in $\mathbf{R}$" is defined by replacing s for $\mathfrak{c}$; see footnote 8.

[11] We may drop ":_" when clear from context and instantiate it by $\mathfrak{c}, Q$, or s otherwise.

Core relations are also called *essential* relations.
The properties that we have described so far can be used for a relational data analysis: A set s is definable in terms of available knowledge $\mathbb{R}$ if there is a subset $\mathbf{R} \subseteq \mathbb{R}$ such that $[\![\mathbf{R}]\!]s = s = \langle\!|\mathbf{R}|\!\rangle s$. Should we discover some $R \in \mathbb{R}$ to be redundant (by not being an element of any reduct) we can safely drop any feature f with $\widetilde{f} = R$ and save storage without loss of information. The last case has two important aspects: First, if $\widetilde{f} \subseteq \widetilde{g}$, we can drop g from $\mathbb{F}$. On the other hand, we might lose hierarchical knowledge in this process. And second, if $\widetilde{f} = \widetilde{g} \cap \widetilde{h}$, we can drop f. In this case, the hierarchical information is retained but the origin of the refined classes (that is, the according attributes) is lost.

4 Relation algebra

Usually, *relation algebra* refers to algebras of endorelations; that is, a set of relations $R : s \rightharpoondown s$ for a common domain s. In formal context analysis we consider attributes or information systems whose intuitive readings suggest *heterogenuous* relations $R : s \rightharpoondown t$ where s is the *type* of the relation's domain and t the type of its codomain. Hence, the base set is the set of all types[12] on which binary relations are defined. Such a relation algebra is referred to as an *abstract relation algebra*, [23].

(Abstract) Relation Algebra. Let $\mathfrak{U}$ be a set of types.

1. For two types $s, t \in \mathfrak{U}$, we define $\mathrm{RA}_{(s,t)}$ to be the set of all relations $R : s \rightharpoondown t$ that together form a complete (atomic) Boolean lattice $\langle s \times t, \cap, \cup, {}^{-}, \perp\!\!\!\perp, \top\!\!\!\top, \subseteq \rangle$.
2. For any three types $r, s, t \in \mathfrak{U}$ there is a function $\fatsemi : \mathrm{RA}_{(r,s)} \times \mathrm{RA}_{(s,t)} \to \mathrm{RA}_{(r,t)}$ with $\fatsemi(P, \fatsemi(Q, R)) = \fatsemi(\fatsemi(P, Q), R)$ which is written as $P \fatsemi Q \fatsemi R$. Also, there are unique elements $\mathit{1}_r$ and $\mathit{1}_t$ such that $\mathit{1}_r \fatsemi R = R$ and $R \fatsemi \mathit{1}_t = R$. Alternative, one demands the set $\mathbf{Rel} = \left\{\mathrm{RA}_{(s,t)} : s, t \in \mathfrak{U}\right\}$ to form a category w.r.t. composition $\fatsemi$ and identities $\mathit{1} : s \mapsto s$. So for any three types r, s, t in $\mathfrak{U}$, the following is a valid diagram in $\mathbf{Rel}$:

$$\begin{tikzcd} r \arrow[loop left, "\mathit{1}_r"] \arrow[r, "\mathfrak{P}"] \arrow[rr, bend right, "\mathfrak{R} := \mathfrak{P} \fatsemi \mathfrak{Q}"'] & s \arrow[loop above, "\mathit{1}_s"] \arrow[r, "\mathfrak{Q}"] & t \arrow[loop right, "\mathit{1}_t"] \end{tikzcd}$$

3. For any $\mathrm{RA}_{(s,t)}$ there is a function $\smile : \mathrm{RA}_{(s,t)} \to \mathrm{RA}_{(t,s)}$ mapping any relation $R \in \mathrm{RA}_{(s,t)}$ on $R^{\smile} \in \mathrm{RA}_{(t,s)}$.
4. The laws of (ordinary) relation algebra (over endorelations) are preserved, which can be expressed expressed as follows:

$$\begin{aligned} R \neq \perp\!\!\!\perp \iff & \top\!\!\!\top \fatsemi R \fatsemi \top\!\!\!\top = \top\!\!\!\top && \text{(Tarski)} \\ P \fatsemi Q \subseteq R \iff & \overline{R} \fatsemi Q^{\smile} \subseteq \overline{P} \iff P^{\smile} \fatsemi \overline{R} \subseteq \overline{Q} && \text{(Schröder)} \end{aligned} \tag{59}$$

   where we implicitely assume type consistency throughout relation composition.

[12] We assume types to be disjoint; i.e. *integer* $\ni \mathtt{1} \neq \mathtt{1} \in$ *real*.

In many proof, we make extensive use of the *Schröder-Equivalences*:

$$P ; Q \subseteq R \iff \overline{R} ; Q^{\smile} \subseteq \overline{P} \iff P^{\smile} ; \overline{R} \subseteq \overline{Q}. \tag{60}$$

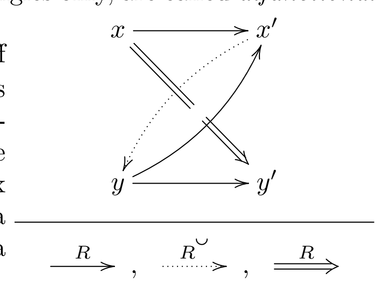
For further details, see [12] and [22]. Also, in every **RA**, the *Dedekind-Rule* is valid:

$$P ; Q \cap R \subseteq (P \cap R ; Q^{\smile}) ; (Q \cap P^{\smile} ; R) . \tag{61}$$

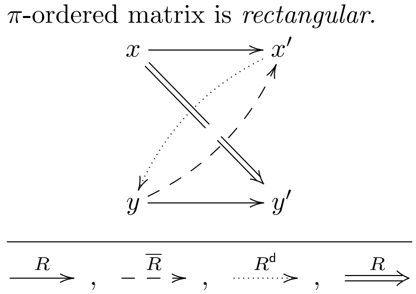
As we shall see, there are several important relation properties that we need to express certain characteristics of information system and which describe particular traits of concept formation, definition and analysis.

4.1 Difunctionality and rectangles

As already pointed out in figure 2, concepts form *rectangles* in heterogeneous binary relations. Relations R that consist of rectangles only, are called *difunctional* relations:



A relation $R : s \rightharpoondown t$ is called *difunctional*, iff $R ; R^{\smile} ; R \subseteq R$.[13] Pointwise speaking, it means that for any two objects $x, y \in s$ sharing a common image $z \in |R\rangle\, x \cap |R\rangle\, y$ their entire image sets coincide: $|R\rangle\, x = |R\rangle\, y$. In terms of matrix representations, R is difunctional, iff there is a permutation $\pi : (s \times t) \to (\mathbb{N}_0 \times \mathbb{N}_0)$ such that a π-ordered matrix is *rectangular*.



A relation $R : s \rightharpoondown t$ is said to have the *Ferrer's property*, iff $R ; R^{\mathsf{d}} ; R \subseteq R$. Pointwise speaking, it means that for any two objects $x, y \in s$ sharing a common image $z \in x.R \cap y.R \in t$ one image set is a subset of the other: $x.R \subseteq y.R$ or $y.R \subseteq x.R$. In terms of matrix representations, R is a Ferrer's relation, iff there is a permutation $\pi : (s \times t) \to (\mathbb{N}_0 \times \mathbb{N}_0)$ such that a π-ordered matrix forms a *staircase*.

If $s = t$, R is called a *biorder*. Using the pointfree definition of difunctionality, we call a context to be a *difunctional context*, iff

$$|\mathtt{P}\rangle\, \langle \mathtt{P}|\, |\mathtt{P}\rangle\, s \subseteq |\mathtt{P}\rangle\, s \text{ for every } s \subseteq \mathcal{U}. \tag{62}$$

As a consequence, if $\mathtt{P}$ is difunctional, then

$$Con(\mathfrak{K}) = \{\langle s, |\mathtt{P}]\, s\rangle : s \subseteq \mathcal{U}\} \cup \{\langle [\mathtt{P}|\, \mathbf{A}, \mathbf{A}\rangle : \mathbf{A} \subseteq \mathbb{A}\}, \tag{63}$$

that is, $\langle s, |\mathtt{P}]\, s\rangle$ and $\langle [\mathtt{P}|\, \mathbf{A}, \mathbf{A}\rangle$ are concepts for every $s \subseteq \mathcal{U}$ and every $\mathbf{A} \subseteq \mathbb{A}$.

---

[13] Note the similarity to equation (14), $|R]\, s = |R]\, [R|\, |R]\, s$.

Proof (Equation 63). Assume $\mathtt{P}$ to be difunctional; i.e. $|\mathtt{P}\rangle\,\langle\mathtt{P}|\,|\mathtt{P}\rangle\,s \subseteq |\mathtt{P}\rangle\,s$.
"$\supseteq$". We show that $\langle s, |P]\,s\rangle$ is a concept. First,

$$|\mathtt{P}]\,s = |\mathtt{P}]\,[\mathtt{P}|\,|\mathtt{P}]\,s \text{ is true by equation 14.} \tag{64}$$

Second, we need to show that $[\mathtt{P}|\,|\mathtt{P}]\,s = s$:

$$[\mathtt{P}|\,|\mathtt{P}]\,s \overset{\text{weakening}}{\subseteq} \langle\mathtt{P}|\,|\mathtt{P}]\,s \overset{\text{weakening}}{\subseteq} \langle\mathtt{P}|\,|\mathtt{P}\rangle\,s \overset{\text{iso,dif}}{\subseteq} s \tag{65}$$

The reverse direction requires us to assume $|\mathtt{P}\rangle\,s \neq \emptyset$. Then, in immediately follows that $s \subseteq [\mathtt{P}|\,|\mathtt{P}]\,s$ by equation 12. The proof for $\langle[\mathtt{P}|\,\mathbf{A}, \mathbf{A}\rangle$ is basically the same.
"$\subseteq$". Let $\langle s, \mathbf{A}\rangle \in \mathit{Con}(\mathfrak{K})$. Then by definition

$$|\mathtt{P}]\,s = \mathbf{A} \text{ and } s = [\mathtt{P}|\,\mathbf{A} \tag{66}$$

such that

$$\langle s, \mathbf{A}\rangle = \langle s, |\mathtt{P}]\,s\rangle = \langle[\mathtt{P}|\,\mathbf{A}, \mathbf{A}\rangle \in \{\langle s, |\mathtt{P}]\,s\rangle : s \subseteq \mathcal{U}\} \cup \{\langle[\mathtt{P}|\,\mathbf{A}, \mathbf{A}\rangle : \mathbf{A} \subseteq \mathbb{A}\}\,.$$

This completes the proof. $\square$

The rôle of rectangles and so-called fringes in applied concept approximation has also been studied extensively, e.g. [21, 9]. Difunctionality also allows us to define equivalences in an elegant way: R is an equivalence, iff it is reflexive and difunctional (see [15]), iff it is a symmetric preorder:

$$\begin{tikzcd}
\mathrm{RTS} \arrow[rr, Leftrightarrow, "(1)"] \arrow[dr, Leftrightarrow, "(2)"'] & & \mathrm{RD} \arrow[dl, Leftrightarrow, "(3)"] \\
& \mathrm{SP} &
\end{tikzcd}$$

where

R means $\mathit{1} \subseteq R$	T means $R \fatsemi R \subseteq R$
D means $R \fatsemi R^{\smile} \fatsemi R \subseteq R$	S means $R^{\smile} = R$

and P means $\mathrm{T} \wedge \mathrm{R}$. (1) is well known, e.g. [15].
(2). Assume RTS. Then, S trivially is true and TS imply P. Hence, the top-down implication on the left side is valid.
(3). Assume SP. Since P equals TR, R is trivially implied. D is shown by symmetry and twice transitivity: $R \fatsemi R^{\smile} \fatsemi R \subseteq R \fatsemi R \fatsemi R \subseteq R \fatsemi R \subseteq R$.

Equivalence relations as edges form graphs that consist of strongly connected components only.

R is called a *f-congruence* relation, if $R : s \rightharpoondown s$ is an equivalence and $f : s \times s \to s$ is a binary operation on s such that R satisfies $xRy \wedge x'Ry' \longrightarrow f(x,y)Rf(x',y')$.
A relation $R : s \rightharpoondown s$ is called a *similarity* relation, iff it is reflexive and symmetric. The definition varies; for example it sometimes seems appropriate drop the symmetry condition (children are considered to be similar to their parents but not vice versa), and sometimes one wants to require (limited) transitivity (such as a public transport zone defined as a set of stations reachable with at most k stops around a center).

4.2 Residuals

Motivation. Given two relations $P : s \rightharpoondown t$ and $Q : s' \rightharpoondown t'$ their (cartesian) product $P \times Q : s \times s' \rightharpoondown t \times t'$ is defined componentwise; the reverse direction is defined through projections $\pi_i : \prod_{j \in \mathbf{n}} R_j \rightharpoondown R_i$ for $i \in \mathbf{n}$. For us, relations sharing domains or codomains are of special interest, for example:

$$
\begin{tikzcd}
 & \mathcal{U} \arrow[dl, "f"'] \arrow[d] \arrow[dr, "g"] & \\
V_f \arrow[r, "f^{\smile}"'] & \mathcal{U} & V_g \arrow[l, "g^{\smile}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & s \arrow[dl, "|\widetilde{f}\rangle\,()"'] \arrow[d] \arrow[dr, "|\widetilde{g}\rangle\,()"] & \\
\mathbf{2}^s \arrow[r, "\mathit{1} \times \top"'] & \mathbf{2}^s \times \mathbf{2}^s & \mathbf{2}^s \arrow[l, "\top \times \mathit{1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{2}^{\mathcal{U} \times \mathbb{F}} \arrow[dl, "|\mathtt{Q}]\,(V)"'] \arrow[d, hook] \arrow[dr, "[\mathtt{Q}|\,(s)"] & \\
\mathbf{2}^{\mathbb{F}} \arrow[r, "[\mathtt{Q}|"'] & \mathbf{2}^{\mathcal{U}} \times \mathbf{2}^{\mathbf{F}} & \mathbf{2}^{\mathcal{U}} \arrow[l, "|\mathtt{Q}]"]
\end{tikzcd}
$$

The left figure illustrates an information $\mathfrak{I}$ with $\mathbf{F} = \{f, g\}$ and corresponding codomains. Converses are well defined and correspond to the (weak) preimage operation. The second figure visualises a more relational, equivalence-focussed interpretation: feature induced equivalences select classes; according operations on the product implement set operations on the equivalence classes (e.g. $\cap : s \times s \to s$). The right diagram depicts a more abstract interpretation of how to identify concept lattices in a given information system: Given U and $\mathbb{F}$ and all possible pairings, strict $\mathtt{Q}$-(pre-) imagesets are determined (w.r.t. feature value and subset restrictions). The converse strict (pre-) imagesets pairwise establish a concept lattice.
A slight rearrangement of diagram layout shows that the underlying structure is that of residual constructions (and, hence, suggests a connection to residuated lattices since both $\wp(\mathbb{F})$ and $\wp(\mathfrak{U})$ form posets and we have a Galois-connection through $[\,|\,|\,]$ and $|\,]\,[\,|$.
These diagrams give rise to the question of defining unique solutions for R. Clearly, $\pi_i : s_1 \times \cdots \times s_n \rightharpoondown s_i$ with $\langle x_1, \ldots, x_n \rangle\, \pi_i x_i$. Its converse $\sigma_i = \pi^{\smile}{}_i$ "*selects*" all 2^{n-1} tuples from the product for a fixed value x_i. Concerning R, we have so-called *fork* and *join* operators (we shall call the latter one *spoon* operator):

$$
\begin{array}{llll}
P \mathrel{-\!\!\in} Q : & s \quad \rightharpoondown t \times t' & \text{with } R = P \mathrel{-\!\!\in} Q & := \; P \fatsemi \pi_1{}^{\smile} \cap Q \fatsemi \pi_2{}^{\smile} \\
P \multimap Q : & s \times s' \rightharpoondown \quad t & \phantom{\text{with }} R = P \multimap Q & := \; \pi_1 \fatsemi P \cap \pi_2 \fatsemi Q
\end{array}
\tag{67}
$$

Also, one might want to know about $R : x \rightharpoondown x'$:

$$
\begin{tikzcd}
 & s \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
t \arrow[rr] & & t'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & t & \\
s \arrow[ur, "P"] \arrow[rr] & & s' \arrow[ul, "Q"']
\end{tikzcd}
$$

Hence, we can decompose R into $R = (\mathit{1}_x \times X_1) \fatsemi (X_2 \times \mathit{1}_{x'})$.

One of the most intriguing questions one is concerned with when dealing with relations is: "When knowing about xPy — what are possible candidates z for xQz or zRy?". Similar questions are: What is the biggest set of constraints that still allows to perform a certain action? What is the smallest set of preconditions that need to be satisfied in order to perform a certain action? And what are the corresponding sets of constraints or postconditions that hold after the action has been performed? All these questions more or less reformulate the concept of *residuals*.

Residuals. For relations P, Q and T with domain and codomain types p, q, r, s we define right and left residuals as follows:

$$\begin{tikzcd}
p \arrow[r, "Q /\!\!/ T"] \arrow[d, "P"'] \arrow[dr, "Q" description] & q \arrow[d, "T"] \\
r \arrow[r, "P \backslash\!\!\backslash Q"'] & s
\end{tikzcd}$$

$$P \backslash\!\!\backslash Q := \overline{P^{\smile} ; \overline{Q}} \tag{68}$$
$$Q /\!\!/ T := \overline{\overline{Q} ; T^{\smile}} \tag{69}$$

Note that both naming and tilt direction varies in literature. The symbol leaning to the *left* denotes the residual for two relations sharing a common domain (hence "*left*").

A common interpretation of residuals is the following:

- The right residual $P \backslash\!\!\backslash R = \overline{P^{\smile} ; \overline{R}}$ is the *biggest* solution for some Q that makes $P ; Q \subseteq R$ true.
- The left residual $P /\!\!/ R = \overline{\overline{P} ; R^{\smile}}$ is the *biggest* solution for some Q that makes $Q ; R \subseteq P$ true.

Residuals follow several useful laws; One can easily verify the following equations:

$$\begin{array}{l} \text{(a)}\ (P \backslash\!\!\backslash Q)^{\smile} = \overline{Q} \backslash\!\!\backslash \overline{P} = Q^{\smile} /\!\!/ P^{\smile} = (\overline{P} /\!\!/ \overline{Q})^{\smile} \\ \text{(b)}\ P \backslash\!\!\backslash Q = \overline{P} /\!\!/ \overline{Q} \qquad \text{(c)}\ P \backslash\!\!\backslash Q^{\mathsf{d}} = P^{\mathsf{d}} /\!\!/ Q\,. \end{array} \tag{70}$$

Talking about sets algebraically, we need to translate them into relations. We therefore refer to a set $s \subseteq \mathcal{U}$ by its characteristic function $\dot{s} : \mathcal{U} \to \mathbf{2}$, the equivalence $\widetilde{s}$ induced by $\dot{s}$ and the subidentity $\mathit{1}_s = \mathit{1}_{\mathcal{U}} \cap s \times s$:

$$x \in s \iff x\dot{s}\mathbf{1} \iff s = |\widetilde{s}\rangle\, x \iff x\mathit{1}_s x. \tag{71}$$

When clear from context, we simply use S for s with an according interpretation to the equivalences above. This way, any set operation can be treated as operations on relations; except for $\cup$ which has to be replaced by $\sqcup$. One interesting aspect that further motivates the use of a residual based treatment is

$$x \in s - t \iff x(\dot{s} \cap \overline{\dot{t}})\mathbf{1} \iff s - t = |\overline{\widetilde{s} \hookrightarrow \widetilde{t}}\rangle\, x \iff x(1_s - \mathit{1}_t)x \tag{72}$$

where $x \hookrightarrow y$ denotes the relative pseudocomplement $\bigsqcup\{z : x \sqcap z \sqsubseteq y\}$. Even though very elegant when it comes down to proofs, point-free relation calculus is not suitable for extensional arguments. For any set s, let $\underset{\cdot}{s}$ be an arbitrary but fixed element of s with $\dot{\underset{\cdot}{s}} = \mathbf{1}$. Let there be a set $\{s_1, \ldots, s_n\}$ of n sets/types s_i. We then call $\left\{\underset{\cdot}{s_i} : 1 \leq i \leq n\right\}$ a *representation set* of the set/type system. Trivially, $x \in s \iff x.\dot{s} = \underset{\cdot}{s}.\dot{s} = \{\mathbf{1}\}$ such that we can reconstruct arbitrary $f \in \mathbb{F}$ from $\widetilde{f}$ and a representation set of V_f as:

$$f(x) = f(\underbrace{[x]_{\widetilde{f}}}_{\cdot}). \tag{73}$$

Hence, the following propositions about membership are equivalent to equation (71):

$$x \in s \iff x.\dot{s} = \underset{\cdot}{s}.\dot{s} = \{\mathbf{1}\} \iff [x]_{\widetilde{\dot{s}}} = s = [\underset{\cdot}{s}]_{\widetilde{\dot{s}}}. \tag{74}$$

Set complementation simply carries over to characteristic functions $\dot{\overline{s}} = \overline{\dot{s}}$.

Residual laws. First, we note that $R;\overline{\dot{s}} = R;\dot{\overline{s}} = \overline{R;\dot{s}}$ for any equivalence R. In particular, we have $\widetilde{s};\dot{s} = \overline{\widetilde{s};\overline{\dot{s}}} = \widetilde{s}\backslash\!\backslash\dot{s}$ which gives us $s = [\underset{\cdot}{s}]_{\widetilde{s}} = \widetilde{s}\backslash\!\backslash\dot{s}\,.\mathbf{1} = \mathbf{1}.\,\dot{s}^{\smile};\widetilde{s}$ and, hence, a pointfree characterisation of equivalence classes. For purposes that will become clear later, we also derive the following equalities for arbitrary equivalences $P, Q : \mathcal{U} \rightharpoondown \mathcal{U}$ using symmetry of P and Q (see also equation (70):

$$P\backslash\!\backslash Q \overset{\backslash\!\backslash}{=\!=} \overline{P;\overline{Q}} \overset{\smile}{=\!=} \overline{(\,\overline{Q};P\,)^{\smile}} \overset{\text{sym } Q}{=\!=} \overline{\overline{Q};P}^{\smile} \overset{/\!/}{=\!=} (\,Q/\!/P\,)^{\smile}, \tag{75}$$

$$\overset{\text{invol}}{=\!=} \overline{\overline{\overline{P}};\overline{Q}^{\smile}}^{\smile} \overset{/\!/,\,\mathsf{d}}{=\!=} P^{\mathsf{d}}/\!/Q^{\mathsf{d}} \overset{\text{sym } Q}{=\!=} \overline{P}/\!/\overline{Q}\,, \tag{76}$$

$$\overset{\smile}{=\!=} \overline{(\,\overline{Q};P\,)^{\smile}} \overset{\text{invol}}{=\!=} \overline{(\,\overline{Q};\overline{\overline{P}}\,)}^{\smile} \overset{\backslash\!\backslash}{=\!=} (\,\overline{Q}\backslash\!\backslash\overline{P}\,)^{\smile}. \tag{77}$$

Finally,

$$\overline{P\backslash\!\backslash Q} \overset{\backslash\!\backslash}{=\!=} \overline{\overline{P^{\smile};\overline{Q}}} \overset{\smile}{=\!=} (\,\overline{Q}^{\smile};P^{\smile}\,)^{\smile} \overset{\text{sym } Q}{=\!=} \overline{\overline{(\,\overline{Q};P^{\smile}\,)}}^{\smile} \overset{/\!/,\,\mathsf{d}}{=\!=} (\,Q/\!/P\,)^{\mathsf{d}}. \tag{78}$$

Also, since $P;P\backslash\!\backslash Q \subseteq Q$, we can verify our formalisation of $[\![\;]\!]$ in equation (83) by instantiating $Q = \dot{s}$ and deducing a tautology by application of the Schröder equivalence:

$$R;R\backslash\!\backslash\dot{s} \subseteq \dot{s} \overset{(59)}{\iff} R^{\smile};\overline{\dot{s}} \subseteq \overline{R\backslash\!\backslash\dot{s}} \overset{\backslash\!\backslash,\,\bar{\bar{\cdot}}}{\iff} R;\overline{\dot{s}} \subseteq R^{\smile};\overline{\dot{s}}\,, \tag{79}$$

which is true (since R is symmetric).

5 A relation algebraic approach to rough sets

The first that comes to mind when discussing the difference between FCA and RST is that in FCA two objects $x, y \in \mathcal{U}$ are different, if there is a feature f for which $f(x) \neq f(y)$. This implies that if $x \in c = \mathit{ext}(\mathbf{c})$ an $y \in d = \mathit{ext}(\mathbf{d})$, $\mathbf{c} \neq \mathbf{d}$. In RST, we know that $x \neq y$, if $[\![\mathbf{R}]\!]\,\{x\} \neq [\![\mathbf{R}]\!]\,\{y\}$. This means there is some $R \in \mathbf{R}$ such that $x\overline{R}y$ — but we do *not* know "why".
First, recall that $\dot{s}$ is defined as the characteristic function of $s \subseteq \mathcal{U}$.

5.1 Rough sets, algebraically

The rough set approximation operators can be defined algebraically by

$$[\![\mathbf{R}]\!]s = [\, R\backslash\!\!\backslash\dot{s} \,|\, \mathbf{1} \text{ and } \langle\!|\mathbf{R}|\!\rangle s = \langle\, R\backslash\!\!\backslash\bar{\dot{s}} \,|\, \mathbf{0}. \tag{80}$$

where $R = \tilde{\tilde{\mathbf{R}}}$. Figure 6 shows an example for computing lower approximations using the residual based definition. Note that for singleton sets $s = \{x\}$ as in

$\dot{c}$	**0** (2)	**1** (2)
□	1	0
■	1	0
▪	0	**1**
●	0	**1**
△	0	**1**
◆	1	0
○	1	0

$\widetilde{shp}$	□ ($\mathcal{U}$)	■	▪	●	△	◆	○
□	*1*	*1*	*1*	0	0	0	0
■	*1*	*1*	*1*	0	0	0	0
▪	*1*	*1*	**1**	0	0	0	0
●	0	0	0	**1**	0	0	*1*
△	0	0	0	0	**1**	0	0
◆	0	0	0	0	0	1	0
○	0	0	0	*1*	0	0	*1*

$\widetilde{shp}\backslash\!\!\backslash\dot{c}$	**0** (2)	**1** (2)
□	0	0
■	0	0
▪	0	0
●	0	0
△	0	**1**
◆	1	0
○	0	0

Hence, $\widetilde{shp}\backslash\!\!\backslash\dot{c} \,.\, \{\mathbf{1}\} = \{△\} = [\![\widetilde{shp}]\!]\,\{▪, ●, △\}$.

Fig. 6. Lower approximations by residuals and characteristic functions

equation (80), $[R|\, s = \langle R|\, s$ (and the same for images).

Proof (Equation 80). See also [14].

$$\begin{aligned}
[\![R]\!]s &= \{x : [x]_R \subseteq s\} \overset{\text{FOL}}{=\!=} \{x : xRy \longrightarrow y\dot{s}\mathbf{1}\} \\
&\overset{\neg\forall}{=\!=} \{x : \neg\exists : \neg(xRy \longrightarrow y\dot{s}\mathbf{1})\} \overset{\text{FOL}}{=\!=} \{x : \neg\exists : \neg(\neg xRy \vee y\dot{s}\mathbf{1})\} \\
&\overset{\text{deM}}{=\!=} \{x : \neg\exists : xRy \wedge y\bar{\dot{s}}\mathbf{1}\} \overset{\neg,\bar{\ }}{=\!=} \overline{\{x : \exists : xRy \wedge y\bar{\dot{s}}\mathbf{1}\}} \\
&\overset{\fatsemi}{=\!=} \overline{\{x : x\, R\fatsemi\bar{\dot{s}}\,\mathbf{1}\}} \overset{\neg,\bar{\ }}{=\!=} \left\{x : x\, \overline{R\fatsemi\bar{\dot{s}}}\,\mathbf{1}\right\} \overset{\text{sym}}{=\!=} \left\{x : x\, \overline{R^{\smile}\fatsemi\bar{\dot{s}}}\,\mathbf{1}\right\} \\
&\overset{\backslash\!\!\backslash}{=\!=}\; R\backslash\!\!\backslash\dot{s} \,.\, \{\mathbf{1}\}\,.
\end{aligned}$$

We can also prove that $x \in [\![R]\!]s \Longleftrightarrow x\, R\backslash\!\!\backslash\dot{s}\,\mathbf{1}$ by splitting the bi-implication:
(1) Let $x \in [\![R]\!]s$. Then,

$$\begin{aligned}
[x]_R \subseteq s &\Longrightarrow xR\underset{\cdot}{s} \\
&\qquad |\!:\ \emptyset \subset \{x\} \subseteq [x]_R,\ \text{and if } [x]_R = \{x\}\ \text{choose } y = x \\
&\Longrightarrow \exists y : xRy \wedge x\dot{s}\mathbf{1} \\
&\qquad |\!:\ \text{Symmetry of } R \text{ and Definition of } \dot{s} \\
&\Longleftrightarrow \exists y : xR^{\smile}y \wedge x\bar{\dot{s}}\mathbf{0} \overset{\fatsemi}{\Longleftrightarrow} x(\, R^{\smile}\fatsemi\bar{\dot{s}}\,)\mathbf{0} \overset{\backslash\!\!\backslash}{\Longleftrightarrow} x\,\overline{R\backslash\!\!\backslash\dot{s}}\,\mathbf{0} \overset{\dot{s}}{\Longleftrightarrow} x\, R\backslash\!\!\backslash\dot{s}\,\mathbf{1}.
\end{aligned}$$

(2) The reverse implication, $x\,R\backslash\!\!\backslash\dot{s}\,\mathbf{1} \Longrightarrow x \in [\![R]\!]s$ is shown by contraposition. We assume $x \notin [\![R]\!]s$. Then,

$$\begin{aligned} x \notin [\![R]\!]s &\overset{[\![\,]\!]}{\Longrightarrow} [x]_R \not\subseteq s \Longleftrightarrow \exists y : y \in [x]_R \wedge y \notin s \\ &\overset{[\,]_R;\dot{s}}{\Longleftrightarrow} \exists y : yRx \wedge y\bar{\dot{s}}\mathbf{1} \overset{;}{\Longleftrightarrow} x\,R^{\smile} ; \bar{\dot{s}}\,\mathbf{1} \overset{\backslash\!\backslash}{\Longleftrightarrow} x\,\overline{R\backslash\!\!\backslash\dot{s}}\,\mathbf{1}. \end{aligned}$$

$\square$

Using the "odd" duality law from equation (9) we can derive the usual interpretation of dual behaviour:

$$[\![\mathbf{R}]\!]\bar{s} \overset{(80)}{=\!=} [\,R\backslash\!\!\backslash\bar{\dot{s}}\,|\,\mathbf{1} \overset{(80)}{=\!=} \langle\,R\backslash\!\!\backslash\bar{\dot{s}}\,|\,\mathbf{1} \overset{\dot{s}}{=\!=} \langle\,\overline{R\backslash\!\!\backslash\bar{\dot{s}}}\,|\,\mathbf{0} \overset{(9)}{=\!=} \overline{\langle\,R\backslash\!\!\backslash\bar{\dot{s}}\,|\,\mathbf{0}} \overset{(80)}{=\!=} \overline{\langle\!\!|\mathbf{R}|\!\!\rangle s}. \qquad (81)$$

Also, the duality is reflected by the residual law in equation (70):

$$\overline{R\backslash\!\!\backslash\overline{S}} = (\,S\backslash\!\!\backslash\overline{R}\,)^{\mathsf{d}} = (\,S^{\mathsf{d}}/\!\!/R\,)^{\mathsf{d}}.$$

where S can be chosen to be $\mathit{1}_s, \dot{s}$ or $\tilde{s} \cap \mathit{1}_s$. We know that $x \in [\![\mathbf{R}]\!]s$, if for all $R \in \mathbf{R}$, $[x]_R \subseteq s$. Hence, $xR\underset{\cdot}{s} \Longleftrightarrow x\,R;\tilde{s}\,\underset{\cdot}{s} \Longleftrightarrow x\,R;\dot{s}\,\mathbf{1}$. Then,

$$[\![\mathbf{R}]\!]s \overset{(a)}{=\!=} R\backslash\!\!\backslash\dot{s}\,.\,\{\mathbf{1}\} \overset{(b)}{=\!=} R\backslash\!\!\backslash\tilde{\dot{s}}\,.\,\{\underset{\cdot}{s}\} \overset{(c)}{=\!=} R\backslash\!\!\backslash\tilde{\dot{s}}.\mathcal{U} \cap s. \qquad (82)$$

The proof can be sketched in just one line (recall that $[\![\mathbf{R}]\!]s = \Big\{y : [y]_{\tilde{\tilde{\mathbf{R}}}} \subseteq s\Big\}$, so if $x \in [\![R]\!]s$, it must hold that $[x]_R \subseteq s$ in particular): So, if $x \in [\![R]\!]s$, then

$$\forall y : xRy \to y \in s \overset{\mathrm{FOL}}{\Longleftrightarrow} \neg\big(xRy \wedge y\bar{\dot{s}}\mathbf{1}\big) \overset{;}{\Longleftrightarrow} x\,\overline{R^{\smile};\bar{\dot{s}}}\,\mathbf{1} \overset{\backslash\!\backslash}{\Longleftrightarrow} x\,R\backslash\!\!\backslash\dot{s}\,\mathbf{1}. \qquad (83)$$

Therefore, we may denote $[\![\mathbf{R}]\!]s$ as $\mathbf{R}\backslash\!\!\backslash S : \mathcal{U} \rightharpoondown \mathcal{U}$ which is short for $\tilde{\tilde{\mathbf{R}}}\backslash\!\!\backslash(\,\mathit{1}_s;\tilde{s}\,)$. Equivalently, we have $[\![\mathbf{R}]\!]\bar{s} = \tilde{\tilde{\mathbf{R}}}\backslash\!\!\backslash\bar{\dot{s}}.\mathbf{1} = \mathbf{0}.(\,\tilde{\tilde{\mathbf{R}}}\backslash\!\!\backslash\dot{s}\,)^{\mathsf{d}} = \overline{\langle\!\!|\mathbf{R}|\!\!\rangle s}$, which reflects the bi-intuitionistic reading suggested by equation (47).
The fact that residuals allow us to express smalles/largest pre-/postconditions, also allows us to define

$$\langle\,R\backslash\!\!\backslash\dot{s}\,|\,\mathbf{1} = \max\{s' : |\dot{s}\rangle\,|R\rangle\,s' = \{\mathbf{1}\}\} \qquad (84)$$

$$\langle\,R\backslash\!\!\backslash\bar{\dot{s}}\,|\,\mathbf{0} = \max\{s' : |\dot{s}\rangle\,|R\rangle\,s' \supseteq \{\mathbf{1}\}\} \qquad (85)$$

Equivalently, $\langle\!\!|R|\!\!\rangle s = \overline{\min\{s' : |\dot{s}\rangle\,|R\rangle\,s' = \{\mathbf{0}\}\}}$. By using the duality expressed above we conclude that the lower and upper approximations are biggest solutions such that

$$\mathit{1}_{[\![R]\!]s} = R\backslash\!\!\backslash s \quad \text{implies} \quad R;\mathit{1}_{[\![R]\!]s} \subseteq s \qquad (86)$$

$$\mathit{1}_{\langle\!|R|\!\rangle s} = (\,s^{\mathsf{d}}/\!\!/R\,)^{\mathsf{d}} \quad \text{implies} \quad \mathit{1}_{[\![R]\!]\bar{s}};R \supseteq \bar{s}. \qquad (87)$$

In allusion to the diagram illustrating the residual definition in equation (68), rough approximations can be displayed as

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\dot{s}/\!\!/R"] \arrow[d, "|R\rangle"'] \arrow[dr, "\dot{s}"] & \mathbf{2} \arrow[d, "-"] \\
\mathcal{U} \arrow[r, "R\backslash\!\!\backslash\dot{s}"'] & \mathbf{2}
\end{tikzcd}$$

Elements $x \in \mathcal{U}$ are *R-identifiable*, iff $\{x\}$ is *R-definable* iff $[\![R]\!]s = s \iff \langle\!|R|\!\rangle s = s$. Again, an algebraic representation offers a very elegant interpretation:

$$R\backslash\!\!\backslash\dot{s} = \dot{s} \Longrightarrow R\fatsemi\dot{s} \subseteq \dot{s} \wedge \overline{\dot{s}} \subseteq R\fatsemi\overline{\dot{s}} \tag{88}$$

$$\overline{R\backslash\!\!\backslash\overline{\dot{s}}} = \dot{s} \Longrightarrow R\fatsemi\dot{s} \subseteq \dot{s} \wedge R\fatsemi\overline{\dot{s}} \subseteq \overline{\dot{s}} \tag{89}$$

which, together, means $R\fatsemi\overline{\dot{s}} = \overline{\dot{s}}$. By application of the Schröder rule, $R\fatsemi\dot{s} = \dot{s}$ and by antitony of complementation $R\fatsemi\overline{\dot{s}} = \overline{R\fatsemi\dot{s}}$. Hence, $R\backslash\!\!\backslash\dot{s} = \dot{s} \iff \dot{s} = \overline{R\backslash\!\!\backslash\overline{\dot{s}}}$.

5.2 Classifications, positive regions, implication

Just as identifiability of objects is a special case of definability of sets, single sets are special cases of classifications. Recall that $\mathfrak{c} = \mathcal{U}/\widetilde{f} = \{\{x : f(x) = v\} : v \in \mathrm{dom}\,(f)\}$. Then,

$$[\![\mathbf{R}]\!]\mathcal{U}/\widetilde{f} = \bigcup_{v \in \mathrm{dom}(f)} \{\langle\, R\backslash\!\!\backslash f \,|\, v\}\,. \tag{90}$$

This characterisation is a 1:1-correspondence to the purely feature- and domain operator-based definition in equations (51-52). Since $\widetilde{f}$ is an equivalence we may generalize to *arbitrary* equivalences Q to express the lower R-approximation of a relation Q, that is, R-positive regions of Q:

$$\begin{tikzcd}
& \mathcal{U} \arrow[dl, "\widetilde{\widetilde{\mathbf{R}}}"'] \arrow[dr, "Q"] & \\
\mathcal{U} \arrow[rr, "\widetilde{\widetilde{\mathbf{R}}}\backslash\!\!\backslash Q", "{[\![\mathbf{R} \lessdot Q]\!]}"'] & & \mathcal{U}
\end{tikzcd}$$

$$[\![R \lessdot Q]\!]\mathcal{U} = \bigcup_{x \in \mathcal{U}} \{\langle\, R\backslash\!\!\backslash Q \,|\, [x]_Q\} \tag{91}$$

$$= \langle\, R\backslash\!\!\backslash Q \,|\, \mathcal{U}. \tag{92}$$

In other words, the $\mathbf{R}$-positive region with respect to Q coincides with the preimageset of the right residual. Again, it turns out that

$$[\![\mathbf{R}]\!]s = [\![\mathbf{R} \lessdot \widetilde{s}]\!]s = \langle\, R\backslash\!\!\backslash\widetilde{s} \,|\, \mathcal{U} \cap s = \langle\, R\backslash\!\!\backslash\dot{s} \,|\, \mathbf{1}. \tag{93}$$

First, instead of defining the lower set approximation algebraically, we start off with positive regions and implication since we have seen that the basic operations $[\![\ ]\!]$ and $\langle\!|\ |\!\rangle$ can be considered special cases of the more abstract constructions. The $\mathbf{P}$-positive region for $\mathbf{Q}$ on $\mathfrak{c} = \mathcal{U}/Q$ is the flat union over all positive regions of $s \in \mathfrak{c}$. Hence, as already pointed out, the $\mathbf{P}$-positive region for $\mathbf{Q}$ on s is the same as $[\![\mathbf{P} \lessdot \mathbf{Q}]\!]\mathcal{U} \cap s$:

$$[\![\mathbf{P} \lessdot \mathbf{Q}]\!]s = (|\, \widetilde{\widetilde{\mathbf{P}}}\backslash\!\!\backslash\widetilde{\widetilde{\mathbf{Q}}}\,\rangle\, s) \cap s. \tag{94}$$

Using appropriate relational representations S of s, we show that $(P\backslash\!\!\backslash Q);S$ equals $[\![P \lessdot Q]\!]s$ (or, as a special case according to equation (93), $(R\backslash\!\!\backslash S;S = [\![R]\!]s)$:

$$
\begin{aligned}
|\,P\backslash\!\!\backslash Q\,\rangle\, s \cap s \overset{\backslash\!\backslash}{=\!=} & \ |\,\overline{P^{\smile};\overline{Q}}\,\rangle\, s \cap s \\
& |\!:\ \text{Change to pointwise notation; translate } \cap s \text{ into } y \in s \\
=\!= & \ \Big\{ y \in s : \exists x \in s : x\, \overline{P^{\smile};\overline{Q}}\, y \Big\} \\
\overset{\text{Logic}}{=\!=} & \ \Big\{ y \in s : \exists x \in s : \neg(x\, P^{\smile};\overline{Q}\, y) \Big\} \\
\overset{;}{=\!=} & \ \Big\{ y \in s : \exists x \in s : \neg(\exists z : xP^{\smile}z \wedge z\overline{Q}y) \Big\} \\
\overset{-}{=\!=} & \ \Big\{ y \in s : \exists x \in s : \neg(\exists z : xP^{\smile}z \wedge \neg(zQy)) \Big\} \\
\overset{\neg\exists}{=\!=} & \ \Big\{ y \in s : \exists x \in s : (\forall z : \neg(xP^{\smile}z \wedge \neg(zQy))) \Big\} \\
\overset{\text{deM}}{=\!=} & \ \Big\{ y \in s : \exists x \in s : (\forall z : (x\overline{P^{\smile}}z \vee zQy)) \Big\} \\
\overset{\longrightarrow}{=\!=} & \ \Big\{ y \in s : \exists x \in s : (\forall z : (xP^{\smile}z \longrightarrow zQy)) \Big\} \\
& |\!:\ P, Q \text{ are indiscernability relations, i.e. equivalences.} \\
=\!= & \ \{ y \in s : \exists x \in s : (\forall z : (z \in [x]_P \longrightarrow z \in [y]_Q)) \} \\
=\!= & \ \{ y \in s : \exists x \in s : [x]_P \subseteq [y]_Q \}
\end{aligned}
$$

A pointwise analysis of equation (91) reveals

$$\bigcup_{c_i \in s/Q} \langle \tilde{\tilde{\mathbf{R}}} \backslash\!\!\backslash Q \,|\, \underset{\cdot}{c_i} = \bigcup_{c_i \in s/Q} \overline{\langle \tilde{\tilde{\mathbf{R}}} |\, \langle \dot{c}_i |\, \mathbf{0}} \tag{95}$$

which relates the lower approximation or residual to a minimal ("$\bigcup$") solution for a complemented iterated preimage (which, we shall see coincides with the preimage under the composed relations). We conclude by definition of $\succeq$ that

$$P \overset{R}{\succeq}_S Q \overset{(56)}{\Longleftrightarrow} Q\backslash\!\!\backslash R \subseteq_S P\backslash\!\!\backslash R \Longrightarrow Q \subseteq_S P. \tag{96}$$

for any subset, classification or equivalence S. A more detailed relation algebraic formalisation of rough set theory can be found in [14].

6 An algebraic approach to formal concept analysis

The connection between FCA and relation algebra is well known, see [19, 20]. However, residuals were mostly interpreted in their "computational meaning" as $\overline{P^{\smile};\overline{Q}}$. Our focus on the abstract interretation of weakest or strongest preconditions is a rather novel approach. It has two big advantages: First, it is a powerful and simple tool to connect FCA and RST and, second, it paves the way for treating FCA in Kleene algebra.

$\mathfrak{I}$	$\mathbb{F}$			
	col	*shp*	*edg*	*siz*
□	w	square	4	S
■	b	square	4	B
▪	b	square	4	S
●	g	circle	1	S
△	w	triangle	3	B
◆	b	diamond	4	S
○	w	circle	1	S

$\mathfrak{I}$	$\mathbf{A} = \mathbf{V}$					
	col		*shp*		*edg*	
	w	b	c	t	3	
□	1	0	0	0	0	
■	0	1	0	0	0	
▪	0	1	0	0	0	
●	0	0	1	0	0	
△	1	0	0	1	1	
◆	0	1	0	0	0	
○	1	0	1	0	0	

where $\mathbf{F} = \{col, shp, edg\}$ and $\mathbf{V} = \{\{\mathtt{b}, \mathtt{w}\}, \{\mathtt{c}, \mathtt{t}\}, \{3\}\}$. Note that by naming convention (see footnote 14), we may drop $\mathbf{F}$ and simply use $\mathbf{V}$ to refer to the attributes.

Fig. 7. Fanning out value restricted features

As already outlined in section 2.2, feature based concepts are more a notational than a paradigmatic different version from attribute based concepts. Where $\langle s, \mathbf{A} \rangle$ is a pre-/semi-/concept based on attributes $f : \mathcal{U} \to \mathbf{2}$ and $x \mathtt{P} f \iff f(x) = \mathbf{1}$, feature based concepts $\langle s, \mathbf{F} \rangle$ simply extend the domain of the attributes to arbitrary sets. In section 1.1, equations (2-4) we already explained the procedure of fanning out features into attributes; and in sections 2.2 and 2.2 we introduced parametrised images by value restrictions.

Let there be a set of Features $\mathbf{F} \subseteq \mathbb{F}$ and a corresponding set of value set restrictions $\mathbf{V} = \{V : V \subseteq \mathrm{dom}(f)\}$. We still assume the codomains of all features to be disjoint such that from the mere value name we an derive a unique attribute.[14]

6.1 Attributes as feature restrictions

Instead of fanning out a feature f w.r.t. $\mathrm{dom}(f)$ we just consider the attributes V. For example, the attribute based concept in figure 3 corresponds to the feature based concept $\langle \{\blacksquare, \blacklozenge\}, \{\{\mathtt{b}\}, \{4\}, \{\mathtt{S}\}\} \rangle$. Therefore, we may write $\langle s, \mathbf{V} \rangle$ to denote $\langle s, \mathbf{F} \rangle$ with parameter restrictions $\mathbf{V}$ on $\mathbf{F}$.

Since features are functions, a strict preimage analysis is useless: If $f(x) = a$ it follows that $x \mathtt{P} a$ *and* $x \overline{\mathtt{P}} y$ for any possible value y other than a. Hence, $[\mathtt{P}|\, V$ must be empty. Instead, the values in V are interpreted as *alternatives*. For our example we have the following CNF-formula (compare to equation (33)):

$$[\mathtt{P}|\, \mathbf{V} = \quad \{x : col(x) = \mathtt{w} \vee col(x) = \mathtt{b}\} \tag{97}$$

$$\cap \{x : shp(x) = \mathtt{c} \vee shp(x) = \mathtt{t}\} \tag{98}$$

$$\cap \{x : edg(x) = 3\} \tag{99}$$

[14] Vice versa, given an arbitrary V, we always can uniquely identify the corresponding feature f. This assumption allows us to avaoid notational overhead by excessive subscripting or using vectors instead of sets.

Note that all attributes originating from one feature have *non-intersecting* equivalence classes. An example for the partitioning induced by $\mathbf{F} = \{f\} = \{shp\}$ and a value restriction to $\mathbf{V} = \{V\} = \{\{\mathtt{square}, \mathtt{circle}\}\}$ is given in parts (1) and (3) of figure 8. In consequence, for any pair of attributes $f, g \in V$, $\widetilde{f} \cup \widetilde{g}$ is an equivalence relation, too (and so is $\bigcup_{f \in V} \widetilde{f}$). Since we, want to consider disjunctions of value alternatives or even non-functional features, we define

$$\widetilde{V} = \bigsqcup_{f \in V} \widetilde{f} = \left(\bigcup_{f \in V} \widetilde{f} \right)^{*}, \tag{100}$$

just to be on the safe side. By denoting $\widetilde{\mathbf{V}} := \left\{ \widetilde{V} : V \in \mathbf{V} \right\}$ we can now express $\mathbf{F}$, i.e. $\mathbf{V}$, as an induced equivalence:

$$\begin{aligned} x \text{ is } \mathbf{V} - \text{equivalent to } y :\Longleftrightarrow \langle x, y \rangle &\in \bigcap_{V \in \mathbf{V}} \bigsqcup_{f \in V} \widetilde{f} \\ &\overset{(100)}{\Longleftrightarrow} x \left(\bigcap \widetilde{\mathbf{V}} \right) y \\ &\overset{(37)}{\Longleftrightarrow} x \widetilde{\widetilde{\mathbf{V}}} y. \end{aligned} \tag{101}$$

that is, we derived an indiscernability relation from $\mathtt{P}$ which leads to the conclusion that

$$\widetilde{\mathtt{P}} := \widetilde{\widetilde{\mathbb{F}}} \subseteq \widetilde{\widetilde{\mathbf{F}}} \subseteq \widetilde{\widetilde{\mathbf{V}}} \subseteq \widetilde{V} \subseteq \widetilde{f}. \tag{102}$$

The proof is trivial, but the inequation reflects what is usually known as *quantisation*. It describes the loss of information along increasingly coarse equivalences (without class intersections). The fundamental idea behind the algebraic representation is that we now treat *both* sets (extents) as subidentities in $\mathcal{U} \times \mathcal{U}$ *and* value restrictions (intents) as equivalences on $\mathcal{U} \times \mathcal{U}$. Hence, we may now consider an algebraic structure in which we can treat both in a unified way on one and the same base set $\mathcal{U}^2$. Still, extents are different from intents which is why we would like to speak of a *typed* base set, c.f. [10]. In our case, this is not neccessary because any extent $s \subseteq \mathcal{U}$ forms a subidentity $\mathit{1}_s \subseteq \mathit{1}$ and every value restriction $\mathbf{V}$ forms an equivalence $\widetilde{\widetilde{\mathbf{V}}} \supseteq \mathit{1}$. Hence, the only case where type distinction is not unique is $\mathit{1}_s = \mathit{1} = \widetilde{\widetilde{\mathbf{V}}}$.
As we need to have $s \subseteq [\mathtt{Q} | \, \mathbf{F}(\mathbf{V})$, all elements of s need to share values from the respective feature value sets. So for every f_v,

$$\mathcal{U}/\widetilde{f}_v = \{\langle f | \, v : v \in V\} \cup \left\{ \overline{\langle f | \, v} \right\} \tag{103}$$

builds a binary classification and, Since every f_v is an attribute defining a binary partition, sets V refer to sets attributes; their mutual disjunction means that

$$x \widetilde{V} y :\Longleftrightarrow x (\bigsqcup V) y = x (\bigcup V)^{*} y. \tag{104}$$

is the partitioning of $\mathcal{U}$ with by f disregarding the feature values that have been ruled out by V. The distinction between admitted values is irrelevant for the

$$
\begin{array}{lrl}
(1) & \mathcal{U}/\widetilde{shp} & = \{\langle f|\, y : y \in \mathrm{dom}\,(f)\} \\
 & & = \{\langle f|\, y : y \in \{\texttt{square}, \texttt{triangle}, \texttt{diamond}, \texttt{circle}\}\} \\
 & & = \{\{\square, \blacksquare, \blacksquare\}, \{\bullet, \circ\}, \{\triangle\}, \{\blacklozenge\}\} \\
(2) & \mathcal{U}/\widetilde{shp}|_V & = \{\langle f|\, y : y \in V\} \,\dot\cup\, \{\langle f|\, \mathrm{dom}\,(f) - V\} \\
 & & = \{\langle f|\, y : y \in \{\texttt{square}, \texttt{circle}\}\} \,\dot\cup\, \{\langle f|\, y : y \in \{\texttt{triangle}, \texttt{diamond}\}\} \\
 & & = \{\{\square, \blacksquare, \blacksquare\}, \{\bullet, \circ\}\} \cup \{\{\triangle, \blacklozenge\}\} \\
 & & = \{\{\square, \blacksquare, \blacksquare\}, \{\bullet, \circ\}, \{\triangle, \blacklozenge\}\} \\
(3) & \mathcal{U}/\widetilde{V} & = \{\langle f|\, V\} \cup \{\langle f|\, \mathrm{dom}\,(f) - V\} \\
 & & = \{\langle f|\, \{\texttt{square}, \texttt{circle}\}\} \cup \{\langle f|\, \{\texttt{triangle}, \texttt{diamond}\}\} \\
 & & = \{\{\square, \blacksquare, \blacksquare, \bullet, \circ\}\} \cup \{\{\triangle, \blacklozenge\}\} \\
 & & = \{\{\square, \blacksquare, \blacksquare, \bullet, \circ\}, \{\triangle, \blacklozenge\}\}
\end{array}
$$

Fig. 8. Three different kinds of equivalences induced by a value restricted feature.

partitioning since f can take only exactly one value. We therefore consider the partitioning with respect to V and $\overline{V} = \mathrm{dom}\,(f) - V$: It is easy to see, that

$$\mathcal{U}/\tilde{f} \sqsubseteq \mathcal{U}/\tilde{f}_V \sqsubseteq \mathcal{U}/\widetilde{V} \tag{105}$$

where $\sqsubseteq$ is the refinement relation: A partition s/P refines a partition s/Q, iff P is a subset of Q, iff every P-class is a subset of a Q-class. Using sets $\mathbf{F}$ of features with value restrictions $\mathbf{V}$, we end up with discernibility relations again:

$$\mathcal{U}/\tilde{\tilde{\mathbf{V}}} := \mathcal{U}/\bigcap\left\{\widetilde{V} : V \in \mathbf{V}\right\} \tag{106}$$

6.2 Semi-rings: From sets to tests

The formalisation in terms of residuals requires a quick detour: Luckily, formal contexts form *idempotent semirings* $\left\langle \mathcal{U}^2, \sqcup, ⨾, \emptyset, \mathit{1} \right\rangle$ with an addition $\sqcup$ and multiplication $⨾$ (see [3].

Theorem. Representing sets and sets of (value-restricted) features as relations and equivalences,

$$\langle \mathcal{U}, \sqcup, ⨾, \emptyset, \mathit{1} \rangle \text{ forms an idempotent semiring.} \tag{107}$$

Proof. The proof is trivial for $\langle \mathcal{U}, \cup, ⨾, \emptyset, \mathit{1}_U \rangle$. $\langle \mathcal{U}, \cup \rangle$ forms a commutative monoid with neutral element $\emptyset$ with idempotent addition $\cup$. $\langle \mathcal{U}, ⨾ \rangle$ is a monoid with neutral element $\mathit{1}$ and the additive unit $\emptyset$ is a multiplicative annihilator. Finally, $⨾$ distributes over $\cup$.

For $\langle \mathcal{U}, \sqcup, ⨾, \emptyset, \mathit{1}_U \rangle$ we examine $\sqcup$ on the set of equivalences and subidentites on $\mathcal{U}$. By equations (29) and (38), $\langle \mathcal{U}, \sqcup \rangle$ forms a complete lattice, hence an upper semi-lattice, hence an idempotent commutative semiring. $\square$

The proof basically shows that contexts correspond to the canonical semiring of binary relations. Should we require the set of relations to be a closed set of equivalences, we need to replace $\cup$ by $\sqcup$ which has been defined using the Kleene-* in equation (39).
We show a few representative properties (P, Q, R denote equivalences, s, t, v subidentities, S, T, V set induced equivalences, X, Y, Z equivalences or subidentites; note that P, Q, R-propositions also hold for S, T, V):

$$\begin{array}{ll}
X \sqcup Y = (X \cup Y)^* = (Y \cup X)^* = Y \sqcup X & \text{(commutativity)} \\
X ; (Y ; Z) = (X ; Y) ; Z & \text{(associativity)} \\
\emptyset \cup X = X = X \cup \emptyset & \text{(neutrality)} \\
P \sqcup P = (P \cup P)^* = P^* = P & \text{(e-idempotency)} \\
s \sqcup s = s^* = \mathcal{U} \supseteq s & \text{(i-idempotency)} \\
P ; \mathit{1} = P = \mathit{1} ; P & \text{(e-neutrality)} \\
s ; \mathit{1} = s \cap \mathit{1} = s = \mathit{1} \cap s = \mathit{1} ; s & \text{(i-neutrality)} \\
X ; \emptyset = \emptyset = \emptyset ; X & \text{(annihilation)} \\
(s \sqcup t) ; u = \mathcal{U} ; u = u \subseteq \mathcal{U} = s ; u \sqcup t ; u & \text{(i-distributivity II)}
\end{array}$$

Formal contexts are Kleene-algebras (for an introduction, see [10] or [8, 11]) where the $*$-laws are satisfied by all domain elements being $*$-closed and $\sqcup$ as the *-closure of $\cup$. For a more in-depth treatment of domain operators in Kleene-algebras, see [3].
Also, the set of extents forms (as we already know) a Boolean subalgebra. From idempotent semirings, we inherit many useful laws; one being that multiplication of subidentites is a lower bound:

$$\mathit{1}_s ; \mathit{1}_t = \mathit{1}_s \cap \mathit{1}_t \text{ such that we may write } \widetilde{s} ; \widetilde{t} = \widetilde{s} \cap \widetilde{t}. \tag{108}$$

From symmetry of $\cap$ and idempotency it follows that the set of all subidentites forms a complete distributive lattice. The distinction between relations representing sets and equivalences induced by features carries over to an extension of Kleene algebras: We will interpret extents as *tests* and intents as *actions* in so-called Kleene algebras with tests (KAT). KATs, on the other hand are specializations of Kleene algebras with domain operators (KAD) which in turn correspond to our (pre-) image operations. Since intents are now represented as equivalences $\widetilde{\mathbf{V}}$ refining $\widetilde{\mathtt{P}}$ [15], we define for the more general case of $R \subseteq \mathcal{U}^2$ (again, see [3]):

$$\lhd(R) \subseteq s \overset{(a)}{\Longleftrightarrow} R \subseteq s ; R \text{ and } \lhd(R) \subseteq s \overset{(b)}{\Longleftrightarrow} \overline{s} ; R \leq \emptyset. \tag{109}$$

For sets (i.e. subidentites or tests as they are called in KAD),

$$\lhd(s) = s = \langle \widetilde{s} |\, s. \tag{110}$$

This way the domain of R can be defined equationally and pointfree. The prose interpretation of (109 a) is that the domain of R can be defined as the smallest

[15] *Proof.* $\widetilde{\mathtt{P}} \subseteq \widetilde{\widetilde{\mathbf{V}}} \overset{\text{iso}}{\Longrightarrow} \widetilde{\mathtt{P}} ; \widetilde{\widetilde{\mathbf{V}}} \subseteq \widetilde{\widetilde{\mathbf{V}}} ; \widetilde{\widetilde{\mathbf{V}}} \overset{\text{trans.}}{\Longrightarrow} \widetilde{\mathtt{P}} ; \widetilde{\widetilde{\mathbf{V}}} \subseteq \widetilde{\widetilde{\mathbf{V}}}$.

set whose subidentity, when used to restrict the preimage of R, does *not* change the image of R. (109 b) states that its complement is the biggest set which, when taken as domain for R has only an empty result. Accordingly the solution s is called a least (left) preserver for R in (a) and greatest (left) annihilator of R in (b). Using "min" and "max" to denote least or greatest solutions, we get

$$\langle \mathtt{Q}|\,\mathbf{V} = \min\{s : \mathbf{V} \subseteq s ⨾ \mathbf{V}\} = \bigcap\{s : \mathbf{V} \subseteq s ⨾ \mathbf{V}\} \tag{111}$$

$$\overline{\langle \mathtt{Q}|\,\mathbf{V}} = \max\{s : s ⨾ \mathbf{V} \subseteq \emptyset\} = \bigcup\{s : s ⨾ \mathbf{V} \subseteq \emptyset\} \tag{112}$$

assuming that s is a subidentity. A further generalisation by dropping the restriction for s being a subidentity leads us to a residual based characterisation: By definitions (68, 69),

$$\lhd(R) = \mathbb{T} ⨾ R^{\smile} = \overline{\mathbb{T} \backslash\!\backslash R^{\mathsf{d}}} = \overline{\emptyset /\!/ R} \tag{113}$$

as a pointfree version of $\langle \mathtt{P}|\, R$

$$\overline{\lhd(R)} = \overline{\mathbb{T} ⨾ R^{\smile}} = \mathbb{T} \backslash\!\backslash R^{\mathsf{d}} = \emptyset /\!/ R \tag{114}$$

as a pointfree version of $\overline{\langle \mathtt{P}|\, R}$.

The connection between preimage and domain operators can be described by the following derivations: We assume an attribute f and want to determine $s = \{x : f(x) = \mathbf{1}\}$ as the domain of f, i.e. the set of all elements in $\mathcal{U}$ having the attribute f. Then, trivially, $f = \dot{s}$ and, again, we can represent s as 1_s or $\langle f|\,\mathbf{1} = \langle \dot{s}|\,\mathbf{1}$:

$$\lhd(f) = \lhd(\dot{s}) = \langle \dot{s}|\,\mathbf{1} = \langle \overline{\overline{\dot{s}}}|\,\mathbf{1} \tag{115}$$

$$\overset{1}{=} \langle\, \overline{1 ⨾ \overline{\dot{s}}}\,|\,\mathbf{1} \overset{\backslash\!\backslash}{=} \langle\, 1 \backslash\!\backslash \dot{s}\,|\,\mathbf{1}$$

$$\overset{(80)}{=} [\![\{\widetilde{s}\}]\!]\mathcal{U} \cap s \overset{(81)}{=} \overline{\overline{[\![\{\widetilde{s}\}]\!]\mathcal{U}} \cup \overline{s}} \overset{(46)}{=} \overline{\langle\!|\{\widetilde{s}\}|\!\rangle\emptyset \cup \overline{s}}$$

$$\overset{(43)}{=} \overline{\emptyset \cup \overline{s}} = \overline{\overline{s}} = s \overset{\cdots}{=} \overline{\lhd(\overline{f})}. \tag{116}$$

Hence, $\langle \mathtt{P}|\, f = \lhd(f)$. Using the pointfree domain operator definition, we have

$$\lhd(S) = \bigcap\{T : 1_s \subseteq T ⨾ 1_S\} = \bigcap\{T : 1_s \subseteq T \cap 1_s\}$$

$$= \bigcap\{T : T = 1_s\} = 1_s \tag{117}$$

$$\overline{\lhd(S)} = \bigcup \overline{\{T : 1_s \subseteq T ⨾ 1_s\}} = \bigcup\{T : T \cap 1_s \subseteq \emptyset\}$$

$$= \bigcup\{T : T = 1_{\overline{s}}\} = 1_{\overline{s}} \tag{118}$$

Proof (Equation 117).

$$\widetilde{\mathtt{P}} \subseteq \widetilde{\widetilde{\mathbf{V}}} \overset{\text{iso}}{\Longrightarrow} \widetilde{\mathtt{P}} ⨾ \widetilde{\widetilde{\mathbf{V}}} \subseteq \widetilde{\widetilde{\mathbf{V}}} ⨾ \widetilde{\widetilde{\mathbf{V}}} \overset{\text{trans.}}{\Longrightarrow} \widetilde{\mathtt{P}} ⨾ \widetilde{\widetilde{\mathbf{V}}} \subseteq \widetilde{\widetilde{\mathbf{V}}}. \square \tag{119}$$

After this tour de force, we can now examine how to characterise (pre-) concepts.

Replacing $\mathbf{V}$ for R in the above finally leads to which demonstrates its close connection to the description of approximation operators in equations (84, 85). This way, we can beautifully embed domain (i.e. extent operations or operations on $\mathcal{U}$) into our relational calculus on the equivalence classes defined by $\tilde{\tilde{\mathbf{V}}}$.
The next important law is required to describe the preimage of a set of **attribute** restrictions. Recall that attributes as binary features correspond to sets and, hence, to subidentities. Also, subidentities S and T we have $S \fatsemi T = S \cap T = T \cap S = T \fatsemi S$. We state the so called *locality* property of the preimage operator

$$\langle S| \, \langle T| \; = \langle \, S \fatsemi T \,| \text{ such that for } \mathbf{A} = \{A_1, \ldots, A_n\} \tag{120}$$

$$\langle \mathbf{A}| \; := \langle A_1| \, \langle A_2| \cdots \langle A_n| \; = \langle \, A_1 \fatsemi A_2 \fatsemi \cdots \fatsemi A_n \,| \; = \langle \bigcap \mathbf{A}| \; = \langle \tilde{\tilde{\mathbf{A}}}| \, . \tag{121}$$

Using $\langle|$ as a preimage operator, S in equations (111) and (112) is interpreted as a subidenty $\mathit{1}_s$.
Using the definition of $\cup$ through *, the Schröder rule, the "odd" duality and the fact that we can translate sets into equivalences, we may state

$$s \subseteq |\mathtt{P}]\,\mathbf{F} \Longrightarrow (\,|\mathtt{P}]\,\mathbf{F} \backslash\!\backslash S\,)^{\smile} \subseteq |\overline{\mathtt{P}}\rangle\,\mathbf{F} \Longleftrightarrow |\mathtt{P}]\,\mathbf{F} \subseteq (\,|\mathtt{P}]\,\mathbf{F} \backslash\!\backslash S\,)^{\mathsf{d}}. \tag{122}$$

Proof (Equation 122). The proof requires a few relation algebraic transforms that may appear a bit peculiar to the reader not familar with relation calculus. For the validity of the rules used, we refer to [12, 22].
Note that all sets in the proof are interpreted as equivalence relations.

$$\begin{aligned}
s \subseteq [\mathtt{P}|\,\mathbf{F} \quad &\Longleftrightarrow \quad s \cup [\mathtt{P}|\,\mathbf{F} = [\mathtt{P}|\,\mathbf{F} \\
&\overset{\text{iso}}{\Longrightarrow} \quad (s \cup [\mathtt{P}|\,\mathbf{F})^* = ([\mathtt{P}|\,\mathbf{F})^* \\
&\overset{\text{Kleene}}{\Longleftrightarrow} \quad s^* \fatsemi (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s^*\,)^* = ([\mathtt{P}|\,\mathbf{F})^* \\
&\qquad |\text{: Interpretation of } s, [\mathtt{P}|\,\mathbf{F} \text{ as } {}^*\text{-closed equivalences.} \\
&\Longleftrightarrow \quad s \fatsemi (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s\,)^* = [\mathtt{P}|\,\mathbf{F} \\
&\qquad |\text{: Isotony of } {}^* \text{ w.r.t. } \fatsemi \text{; Equality implies subsets} \\
&\Longrightarrow \quad s \fatsemi (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s\,) \subseteq s \fatsemi (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s\,)^* \subseteq [\mathtt{P}|\,\mathbf{F} \\
&\overset{\text{Schröder}}{\Longrightarrow} \quad \overline{[\mathtt{P}|\,\mathbf{F}} \fatsemi (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s\,)^{\smile} \subseteq \overline{s} && (123) \\
&\qquad |\text{: Conv. of equiv; “odd duality”, (9); associativity} \\
&\Longleftrightarrow \quad \langle\overline{\mathtt{P}}|\,\mathbf{F} \fatsemi s \fatsemi [\mathtt{P}|\,\mathbf{F} \subseteq \overline{s} && (124)
\end{aligned}$$

Then, we can further deduce:

$$\begin{aligned}
(123) \quad &\overset{/\!/}{\Longleftrightarrow} \quad \overline{[\mathtt{P}|\,\mathbf{F} /\!/ (\,[\mathtt{P}|\,\mathbf{F} \fatsemi s\,)} \subseteq \overline{s} \Longleftrightarrow s \subseteq [\mathtt{P}|\,\mathbf{F} /\!/ (\,([\mathtt{P}|\,\mathbf{F}) \fatsemi s\,) \\
&\Longleftrightarrow \quad \overline{[\mathtt{P}|\,\mathbf{F}} \fatsemi s \fatsemi [\mathtt{P}|\,\mathbf{F} \subseteq \overline{s} \Longleftrightarrow s \subseteq \overline{\overline{[\mathtt{P}|\,\mathbf{F}} \fatsemi s \fatsemi [\mathtt{P}|\,\mathbf{F}} && (125) \\
(124) \quad &\overset{(123)\to(124)}{\Longleftrightarrow} \quad \overline{[\mathtt{P}|\,\mathbf{F}} \fatsemi \overline{s \backslash\!\backslash \langle\overline{\mathtt{P}}|\,\mathbf{F}} \subseteq \overline{s} \Longleftrightarrow (\langle\overline{\mathtt{P}}|\,\mathbf{F}) \fatsemi s \subseteq s \backslash\!\backslash \langle\overline{\mathtt{P}}|\,\mathbf{F}
\end{aligned}$$

Therefore, we can reformulate $[\mathtt{P}|\,\mathbf{V}$ as being the biggest subset of $\mathcal{U}$ satisfying $\mathbf{V}$. This formalisation gives rise to two further interpretations: First, it corresponds to the definition of relative pseudocomplements and second, by using as a multiplication and transforming the set inclusion $s \subseteq t$ to the equation $s \cup t = t$ based on $\cup$ we have an addition which together with $\fatsemi$ forms an idempotent semiring (i.e. upper semi-lattice).

6.3 An algebraic specification of pre-concepts

According to definition (22),

$$\langle s, \mathbf{A} \rangle \text{ is a preconcept, iff } s \subseteq [\mathtt{P}|\, \mathbf{A} \text{ and } \mathbf{A} \subseteq |\mathtt{P}]\, s \tag{126}$$

We treat the first condition ($s \subseteq [\mathtt{P}|\, \mathbf{A}$) only, because the second one can be inferred by the symmetry of the (pre-) image operators as shown in equations (5) and (7). Recall that $\mathbf{A}$ is a set of attributes $f : \mathcal{U} \to \mathbf{2}$, hence f is "its own characteristic function". According to (117,118), $\langle \mathtt{P}|\, f$ coincides with $\lhd(\mathit{1}_f)$ if we assume $\mathit{1}_f = \mathit{1} \cap (\langle f|\, \mathbf{1} \times \langle f|\, \mathbf{1})$. Therefore,

$$
\begin{aligned}
[\mathtt{P}|\, \mathbf{A} &\overset{(7)}{=\!=} \bigcap_{f \in \mathbf{A}} \langle \mathtt{P}|\, f \overset{(115,116)}{=\!=} \bigcap_{f \in \mathbf{A}} \lhd(f) && (127)\\
&\quad |\text{: All } f \text{ are subidentities; push } \textstyle\bigcap \text{ inside:}\\
&\overset{(108)}{=\!=} \lhd(f_0) \fatsemi \cdots \fatsemi \lhd(f_{n-1})\\
&\quad |\text{: By KAD; see [3], Lemma 4.11 } \textit{(iv), (vii), (viii).} && (128)\\
&=\!= \lhd(f_0 \fatsemi \cdots \fatsemi f_{n-1})\\
&\quad |\text{: All } f \text{ are subidentities; rewrite:}\\
&\overset{(108)}{=\!=} \lhd(f_0 \cap \cdots \cap f_{n-1}) && (129)\\
&=\!= \lhd(\textstyle\bigcap \mathbf{A})\\
&\overset{(113)}{=\!=} \overline{\mathbb{T} \backslash\!\backslash (\textstyle\bigcap \mathbf{A})^{\mathsf{d}}} && (130)\\
&\quad |\text{: Since } \lhd(\mathbb{T}) = \lhd(\mathit{1}), \text{ we translate the residual}\\
\text{by (80)} : \langle\!| \mathit{1} |\!\rangle \textstyle\bigcap \mathbf{A} &=\!= \textstyle\bigcap \mathbf{A} \overset{(45)}{=\!=} [\![\mathit{1}]\!] \textstyle\bigcap \mathbf{A}. && (131)
\end{aligned}
$$

We conclude: Whatever we are able to identify by proving it has certain properties in a formal context, can be distinguished from any object that does not satisfy these properties by the notion of definability in rough set theory.
In sections 2.2 and 6.1 we introduced features (multi-valued attributes) and value restrictions $\mathbf{V}$ that are interpreted as propositional CNF-formulae with feature-value induced attributes: $\mathbf{V} = \{V_0, \ldots, V_m\}$ with $V_i = \{f_0, \ldots, f_{n_i}\}$ describes the set of elements $x \in \mathcal{U}$ satisfying

$$\bigwedge_{j \in \mathbf{m}} \bigvee_{i \in \mathbf{n}_j} f_{j_i}(x) = \mathbf{1}. \tag{132}$$

which is an attribute logic expression equivalent to the set-based expression in equation (97), see [2]. Recall that all attributes result from fanning out the disjunctive value alternatives for features. Therefore:

$$\langle \mathtt{Q}|\, V = \bigcup_{f\in V} \langle \mathtt{Q}|\, f \overset{(115,116)}{=} \bigcup_{f\in V} \lhd(f) \tag{133}$$

$$|\text{: By KAD; see [3], Lemma 4.11 } \textit{(ii)}$$

$$= \lhd(\bigcup_{f\in V} f) =: \lhd(V) =: \mathit{1}_V\,. \tag{134}$$

The disjunction of feature values results in the least subidentity $\mathit{1}_V$ subsuming all feature value induced subidentities. Then,

$$[\mathtt{Q}|\, \mathbf{V} = \bigcap_{V\in\mathbf{V}} \langle \mathtt{Q}|\, V \overset{(134)}{=} \bigcap_{V\in\mathbf{V}} \lhd(\bigcup_{f\in V} f) \tag{135}$$

$$\overset{(134)}{=} \bigcap_{V\in\mathbf{V}} \mathit{1}_V \overset{(128)}{=} \lhd(\mathit{1}_{V_0} \,;\, \cdots \,;\, \mathit{1}_{V_{n-1}}) =: \mathit{1}_{\mathbf{V}} \tag{136}$$

We conclude that, similar to the result in equation (131), value restrictions $\mathbf{V}$ define strict preimages under $\mathtt{Q}$ and any two elements x and y of this set satisfy formula (132).
Finally, we define preconcepts according to equation (126):

$$s \subseteq [\mathtt{P}|\, \mathbf{V} \iff s \subseteq \lhd(\mathbf{V}) \iff s \subseteq [\![\mathbf{V}]\!] \left\{ \underset{\cdot}{V}_0, \ldots, \underset{\cdot}{V}_{n-1} \right\}. \tag{137}$$

The reverse case is, as mentioned at the beginning of this section, ommitted because it follows from the fact that $[R|\, s = |R^{\smile}]\, s$.
While the domain operator $\lhd(\mathbf{V})$ is algebraically the most elegant way to describe concepts, it requires at least preimages and $\mathtt{P}$ to determine a concept's extension. In addition, $\mathtt{P}$ is a very inefficient since sparse and representation of $\mathtt{Q}$. In rough set theory it suffices to keep information about $\tilde{\tilde{\mathbf{V}}}$ and one representative $\underset{\cdot}{V}_i \in \mathcal{U}$ for which there is a $j \in \mathbf{n}_i$ such that f_{i_j} delivers $\mathbf{1}$.

7 Conclusion

The connections between FCA and RST have been studied for a long time from a large variety of points of view. Our contribution is to represent the fundamental constructions in RST and FCA using residuals in relation algebra. Residuals can be interpreted as greatest or least solutions of inequations in ordered structures. Also, both FCA and RST (being set- or lattice theory constructs) have a canonical propositional logic interpretation *plus* upper or lower approximations (as least or greatest super/subsets) and their modal propositional logic counterparts. Residuals form quantifiers ([19] gives a nice combinatoric overview of the 2^8 different meanings of residuals according to complementing the different arguments) or "filters"; in particular, they define preconditions in terms of attributes that must be present or must not be present (or unspecified). This

gives rise to further logical models like bi-intuitionistic logic (recall pseudocomplements $s \hookrightarrow t$, equation (72)) or modal logics. Then, via semi-rings we come to modalities in semi-rings and finally, to Kleene algebra with domain operators and, eventually, back to residuals.
However, the number of questions raised by our presentation exceeds the number of answered questions by far. With the toolset presented here, it seems promising to work on one or more of the following questions:

- The formalisation of RST in RA is nearly complete for we have seen that $[\![P \lessdot R]\!]$ can be mapped on $\langle\, P \backslash\!\!\backslash R \,|$ from which all other concepts can be inferred. However, we are still missing an explicit description of implication, reducts and cores.
- We only gave a very shallow idea of how to formalise FCA in KAD. A more detailed description and analysis is required; especially to analyse all avanced concepts like sub- and supercontexts (and closure), implications and dependencies, etc.
- An interesting "practical" aspect is the following: In RST, the usual question is which relations are the best to describe a set. In machine learning, one would rather like to *construct* a relation which together with a given core creates a reduct that is able to describe the set.
- Speaking of learning and $[\![P \lessdot R]\!]$, it is natural to ask for some dual like $\overline{[\![P \lessdot R]\!]}$. Not surprisingly, we observe some complementary residuals, some common dualities and some "odd" dualities. This question is in the focus of ongoing work.

Finally, even though very ambitious, we know how to specify algorithms in KAD. It would be great or is at least a great motivation to try implementations of RST and FCA algorithms (Skowron's algorithm for finding core relations, [14] or decompositions of concept lattices).

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